On the Communication Complexity of Approximate Pattern Matching

Jakob Nogler[2]

based on joint work with

Tomasz Kociumaka[1] **Philip Wellnitz**[3]

[1]Max Planck Institute for Informatics, SIC ($\rightarrow$ INSAIT)

[2]ETH Zurich

[3]National Institute of Informatics, SOKENDAI

Strings

- A *string* is a sequence of characters from an *alphabet*.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
S	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b

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$S[12]$

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X	b	b	a	a	a	b	b	b	a	b	b	a	a
Y	a	b	a	b	a	b	b	b	a	b	b	a	a

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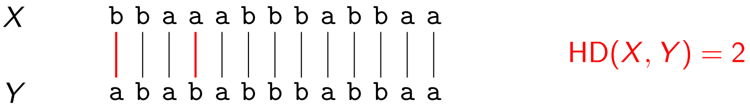

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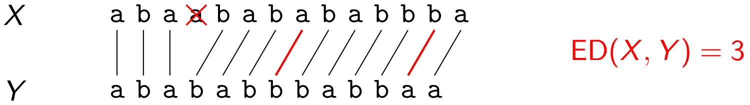

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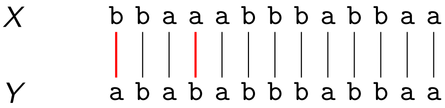

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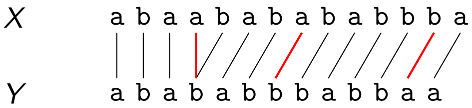

$\mathrm{ED}(X, Y) = 3$

Alignment

Pattern Matching (PM)

Text T, $\|T\| = n$	`a b a a b a b a b a b b b a b b a a a b b b a b b a a`
Pattern P, $\|P\| = m$	`a b a b a b b b a b b a a`

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$k = 2$

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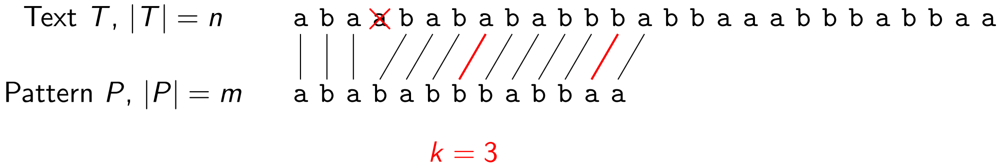

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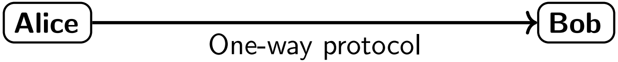
Communication Complexity



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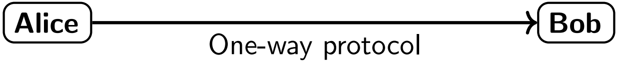

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Text T, Pattern P, Threshold k

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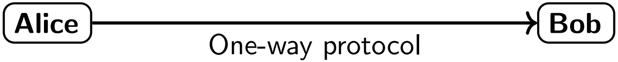

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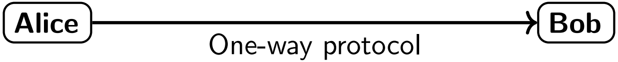

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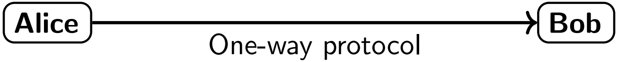

1. Alice receives a PM instance. Text T, Pattern P, Threshold k
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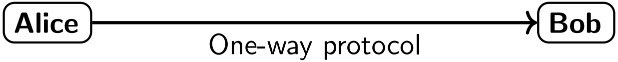

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Set $Occ^E_k(P, T)$

Communication Complexity = “minimum # of machine words to send to Bob”

Example for Exact Pattern Matching

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3. She can send P, T.

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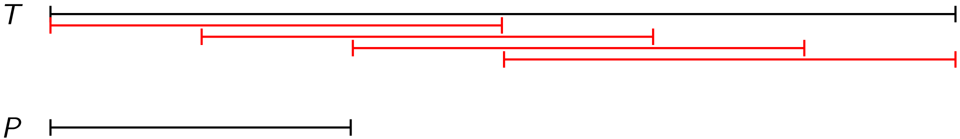

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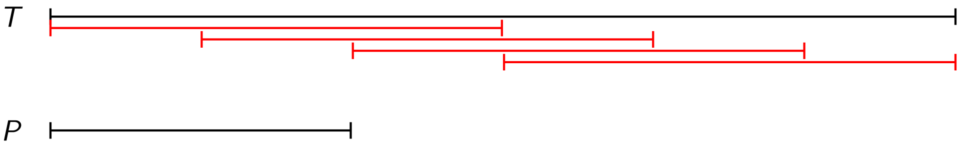

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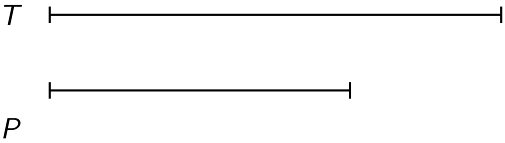

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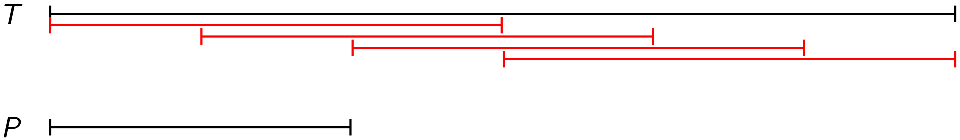

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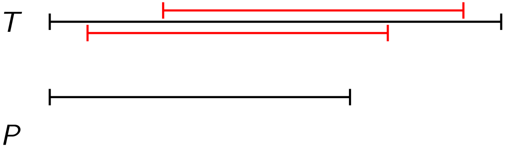

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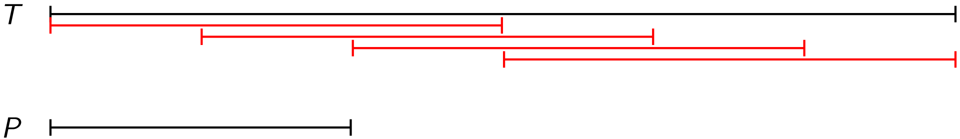

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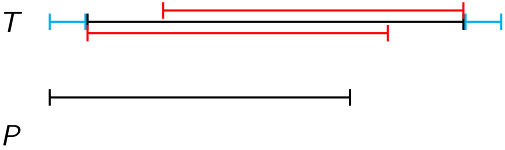

Previous Results

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N.B. In **[CKP19]** Alice sends to Bob $\mathrm{Occ}_k^H(P, T)$ **plus** the *mismatch information* $\mathrm{MI}(x)$ for all $x \in \mathrm{Occ}_k^H(P, T)$, defined as

$$\mathrm{MI}(x) := \{(i, P[i], T[x+i]) \mid i \in [0..m) \text{ such that } P[i] \neq T[x+i]\}.$$

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PM with edits	$\boldsymbol{\mathcal{O}(k \log n)}$	$\boldsymbol{\Omega(k)}$	**[KNW24]**

Exact Pattern Matching

Exact PM

Periodicity Lemma Readapted [FW65]

If $n \leq 3/2 \cdot m$ and $\{0, n-m\} \subseteq \text{Occ}(P, T)$, then $\gcd(\text{Occ}(P, T))$ is a period of T.

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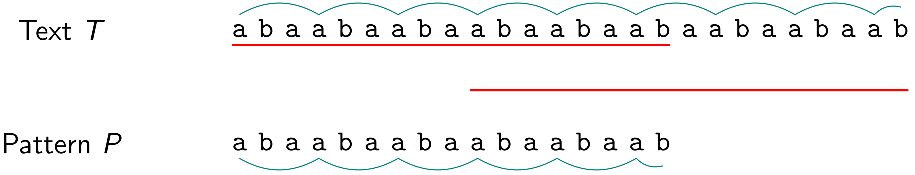

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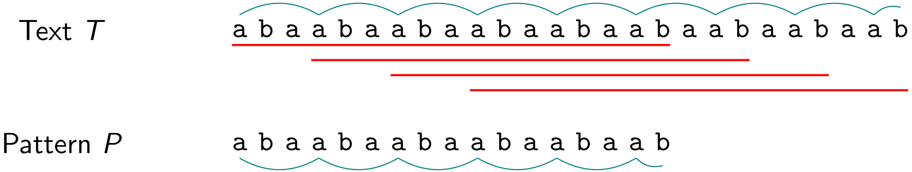

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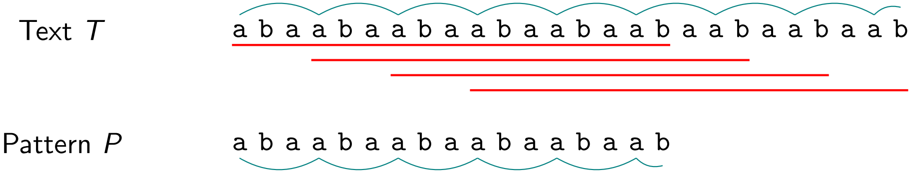

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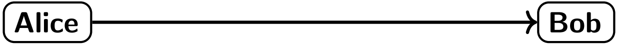
- But for every $x \in \mathrm{Occ}(P,T)$, we have $x \mid g$. Thus, $x \in A$ and $A = \mathrm{Occ}(P,T)$.
- In order to send A to Bob, it suffices that Alice sends two numbers: g and $|A|$.

Pattern Matching with Mismatches

What Alice Sends



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$$\{0, n-m\} \subseteq S \subseteq \mathrm{Occ}_k^H(P, T)$$

s.t. $\gcd(S) = \gcd(\mathrm{Occ}_k^H(P, T))$.

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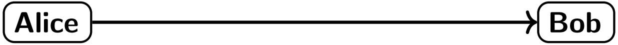

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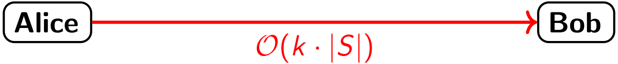

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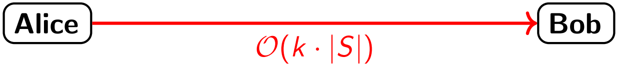

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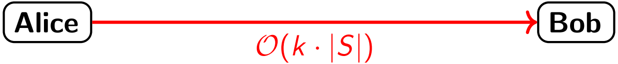
We can choose S s.t. $|S| \leq \mathcal{O}(\log n)$.

What Alice Sends



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s.t. $\gcd(S) = \gcd(\mathrm{Occ}_k^H(P, T))$.

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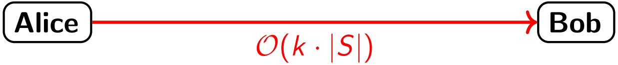

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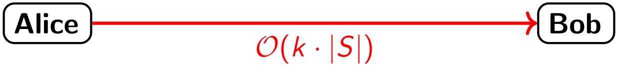
- Construct S iteratively.
- Try to add to S elements from $\mathrm{Occ}_k^H(P, T)$ one by one.

What Alice Sends



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- For each element x either $\gcd(S \cup \{x\}) = \gcd(S)$ or $\gcd(S \cup \{x\}) \leq \gcd(S)/2$.

What Alice Sends

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What Alice Sends (Example)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	a	b	c	b	a	b	c	b	a	b	a	b	a	b	a
P	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	c	b	a	b	a	b	a	b	a						

$$k = 4$$

What Alice Sends (Example)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	a	b	c	b	a	b	c	b	a	b	a	b	a	b	a
P	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	c	b	a	b	a	b	a	b	a						

$$\boldsymbol{k = 4}$$

$$\mathrm{Occ}_k^H(P, T) = \{0\}$$

What Alice Sends (Example)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	a	b	c	b	a	b	c	b	a	b	a	b	a	b	a
P			b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	c	b	a	b	a	b	a	b	a				

$$\boldsymbol{k = 4}$$

$$\mathrm{Occ}_k^H(P, T) = \{0, 2\}$$

What Alice Sends (Example)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	a	b	c	b	a	b	c	b	a	b	a	b	a	b	a
P					b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	c	b	a	b	a	b	a	b	a		

$$\boldsymbol{k = 4}$$

$$\mathrm{Occ}_k^H(P, T) = \{0, 2, 4\}$$

What Alice Sends (Example)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	a	b	c	b	a	b	c	b	a	b	a	b	a	b	a
P							b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	c	b	a	b	a	b	a	b	a

$$\boldsymbol{k = 4}$$

$$\mathrm{Occ}_k^H(P, T) = \{0, 2, 4, 6\}$$

What Alice Sends (Example)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	a	b	c	b	a	b	c	b	a	b	a	b	a	b	a

$$\boldsymbol{k = 4}$$

$$\mathrm{Occ}_k^H(P, T) = \{0, 2, 4, 6\}$$

Alice sends $S = \{0, 2, 6\}$

What Alice Sends (Example)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	a	b	c	b	a	b	c	b	a	b	a	b	a	b	a
P	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	c	b	a	b	a	b	a	b	a						
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23						

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Alice sends $S = \{\mathbf{0}, 2, 6\}$
and
$\{(15, \mathrm{c}, \mathrm{a}), (17, \mathrm{a}, \mathrm{c}), (21, \mathrm{a}, \mathrm{c})\}$

What Alice Sends (Example)

```
      0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29
T     b e b e b e b e b e b  e  b  e  b  a  b  c  b  a  b  c  b  a  b  a  b  a  b  a
          | | | | | | | | |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
P         b e b e b e b e b  e  b  e  b  e  b  c  b  a  b  a  b  a  b  a
          0 1 2 3 4 5 6 7 8  9  10 11 12 13 14 15 16 17 18 19 20 21 22 23
```

$$\boldsymbol{k = 4}$$

$$\mathrm{Occ}_k^H(P, T) = \{0, 2, 4, 6\}$$

Alice sends $S = \{0, \mathbf{2}, 6\}$

and

$\{(15, \mathrm{c}, \mathrm{a}), (17, \mathrm{a}, \mathrm{c}), (21, \mathrm{a}, \mathrm{c})\}$

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What Alice Sends (Example)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	a	b	c	b	a	b	c	b	a	b	a	b	a	b	a
P							b	e	b	e	b	e	b	e	b	e	b	e	b	e	b	c	b	a	b	a	b	a	b	a
							0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23

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$\{(9, \mathrm{e}, \mathrm{a}), (11, \mathrm{e}, \mathrm{c}), (13, \mathrm{e}, \mathrm{a})\}$

What Bob Receives (Example)

Bob receives $S = \{0, 2, 6\}$

and

$\{(15, \text{c}, \text{a}), (17, \text{a}, \text{c}), (21, \text{a}, \text{c})\}\}$

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
P	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?

What Bob Receives (Example)

Bob receives $S = \{0, 2, 6\}$

and

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
T	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	a	?	c	?	a	?	c	?	?	?	?	?	?	?	?

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
P	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	c	?	a	?	?	?	a	?	?

What Bob Receives (Example)

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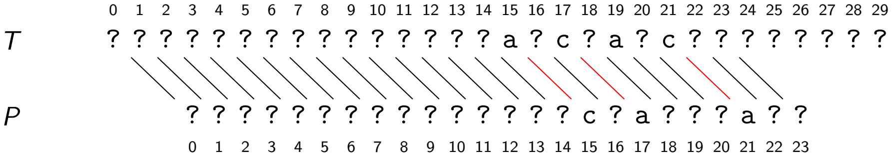

What Bob Receives (Example)

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and

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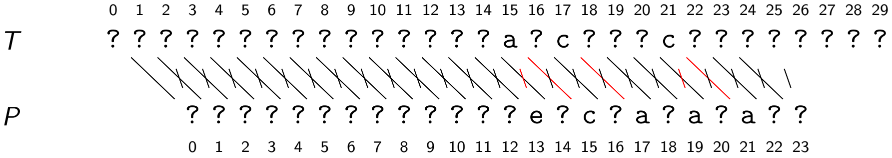

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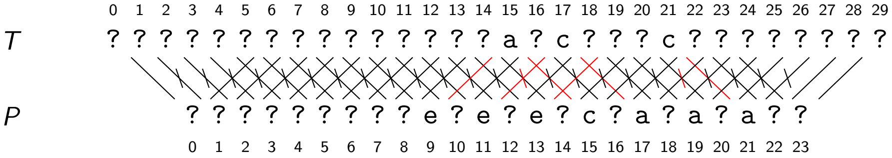
Inference graph $\mathbf{G}_S$



What Bob Receives (Example)

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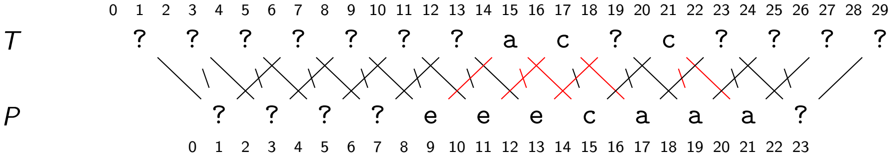

Red connected component (at least one red edge)

What Bob Receives (Example)

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and

$\{(15, \mathrm{c}, \mathrm{a}), (17, \mathrm{a}, \mathrm{c}), (21, \mathrm{a}, \mathrm{c})\}\}$

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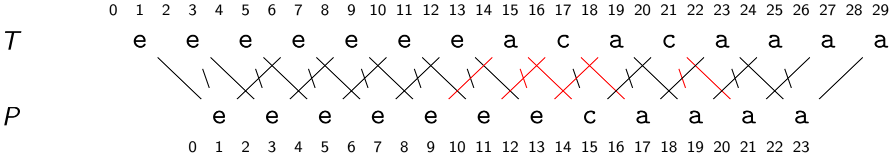

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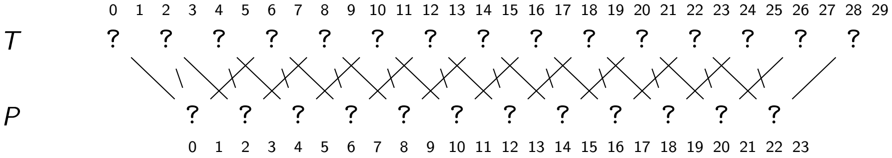
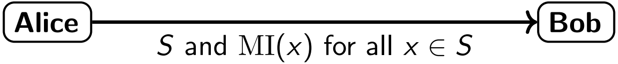


Black connected component (no red edge)

What Bob Receives



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  - $\{t_{i+j}, p_i\} \in E$ for all $i \in S, j \in [0..m)$, edge is red if is a mismatch, otw it is **black**.

The Structure of Connected Components in $\mathbf{G}_S$

Periodicity Lemma Readapted [FW65]

If $n \leq 3/2 \cdot m$ and $\{0, n - m\} \subseteq \mathrm{Occ}(P, T)$, then $\gcd(\mathrm{Occ}(P, T))$ is a period of T.

Lemma

Let $g := \gcd(S)$. Then, $\mathbf{G}_S$ has g connected components. Moreover, the i-th connected component for $i \in [0..g)$ contains

$$\{p_j \mid j \equiv_g i\} \cup \{t_j \mid j \equiv_g i\}.$$

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- For each $x \in S$, we have $x \in \mathrm{Occ}(P^\$, T^\$)$. Thus, $S \subseteq \mathrm{Occ}(P^\$, T^\$)$.

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- As $\{0, n-m\} \subseteq \mathrm{Occ}(P^\$, T^\$)$, we can apply the Apply Periodicity Lemma.
- The period of $P^\$$ and $T^\$$ is at most $\gcd(\mathrm{Occ}(P^\$, T^\$)) \leq g$

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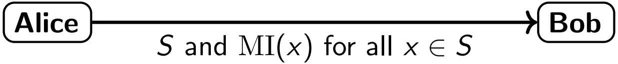
- As $\{0, n-m\} \subseteq \mathrm{Occ}(P^\$, T^\$)$, we can apply the Apply Periodicity Lemma.
- The period of $P^\$$ and $T^\$$ is at most $\gcd(\mathrm{Occ}(P^\$, T^\$)) \leq g$
- $\mathbf{G}_S$ has at most g connected components.

Bob constructs $P^{\#}$ and $T^{\#}$



- Alice selects a subset

  $$\{0, n-m\} \subseteq S \subseteq \mathrm{Occ}_k^H(P, T)$$

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- Bob receives S and $\mathrm{MI}(x)$ for all $x \in S$.
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  - $V = \{t_0, \ldots, t_{n-1}, p_0, \ldots, p_{m-1}\}$, and
  - $\{t_{i+j}, p_j\} \in E$ for all $i \in S, j \in [0..m)$, edge is red if is a mismatch, otw it is **black**.
- Bob construct strings $P^{\#}, T^{\#}$ from P, T by replacing each character contained in a black component in $\mathbf{G}_S$ with a sentinel character (unique to the component the character is contained).
- Bob computes $\mathrm{Occ}_k^H(P^{\#}, T^{\#})$.

Examples for $P^{\#}$ and $T^{\#}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
$T^{\#}$	#	e	#	e	#	e	#	e	#	e	#	e	#	e	#	a	#	c	#	a	#	c	#	a	#	a	#	a	#	a
$P^{\#}$				#	e	#	e	#	e	#	e	#	e	#	e	#	e	#	c	#	a	#	a	#	a	#	a			
				0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23			

$P^{\#}$ and $T^{\#}$ preserve k-mismatch occurrences

Lemma

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  - Thus, p_j and t_{j+i} are contained in the same connected component:
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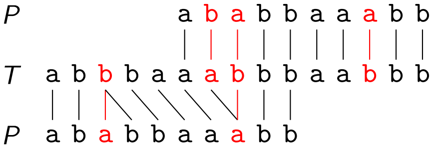
  - As j was arbitrary, $\mathrm{HD}(P^{\#}, T^{\#}[i..i+m)) = \mathrm{HD}(P, T[i..i+m))$ and $i \in \mathrm{Occ}_k^H(P^{\#}, T^{\#})$.

Pattern Matching with Edits

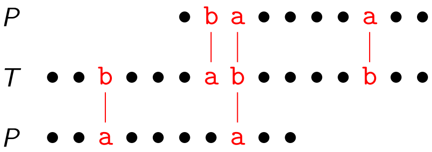
Finding a Period Structure

Suppose Alice takes a set S of alignments of cost at most k,



Finding a Period Structure

Suppose Alice takes a set S of alignments of cost at most k, and sends to Bob only the information about edits.



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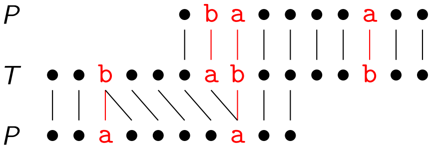

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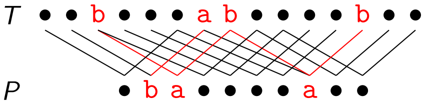

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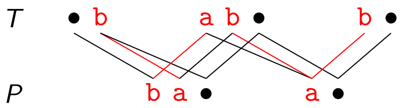

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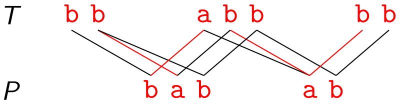

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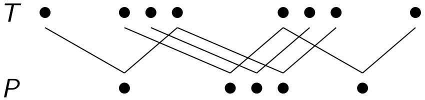

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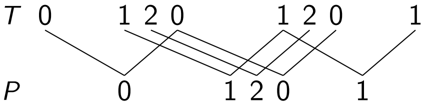

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T 0 1 2 0 1 2 0 1

P 0 1 2 0 1

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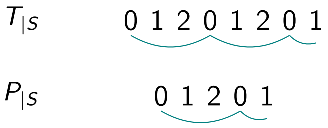

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Mapping Back the Periodic Structure to the Original Strings

$T_{|S}$ **0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5**

$P_{|S}$ **0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5**

Mapping Back the Periodic Structure to the Original Strings

$T_{\|S}$	0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5
T	b 0 1 2 3 4 5 6 7 8 a 0 1 2 3 b 4 5 6 7 8 0 1 2 3 a a 4 5 6 7 8 a 0 1 2 3 4 5 6 7 8 0 1 2 a 3 4 5
$P_{\|S}$	0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5
P	a a 0 1 2 3 4 5 6 7 8 0 1 2 3 b b 4 5 6 7 8 0 1 2 3 a 4 5

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τ_0 τ_1 τ_2 τ_3 τ_4

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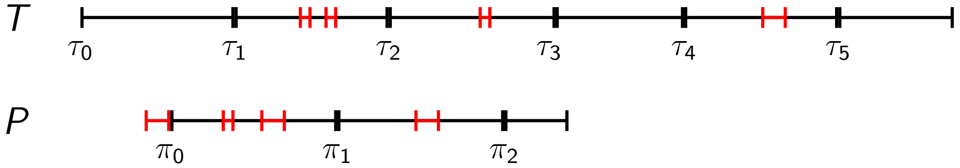
π_0 π_1 π_2

Alignments Covered by S



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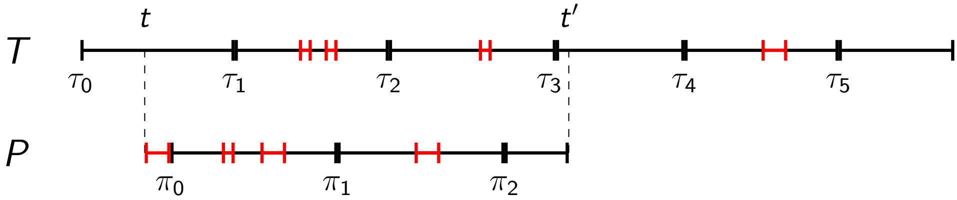

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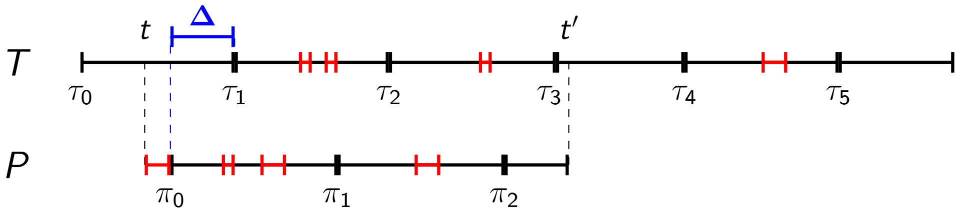

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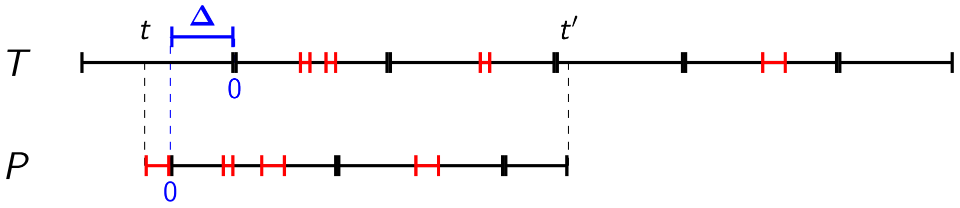

Case $\Delta = \widetilde{\Omega}(k)$ (large)

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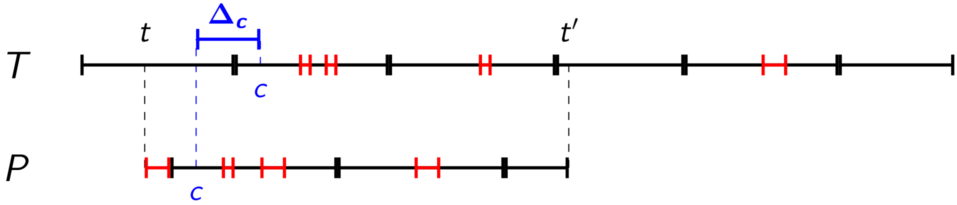

Case $\Delta = \widetilde{\Omega}(k)$ (large) $\implies$ if $\mathcal{X}$ is added to S, the # of black components at least halves.

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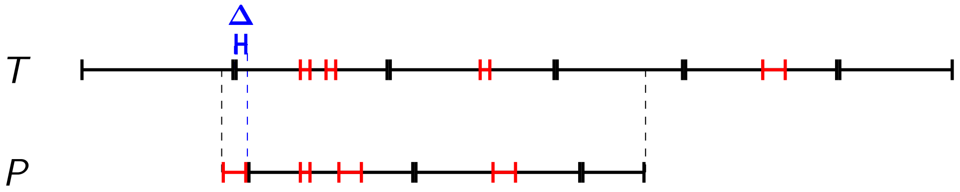

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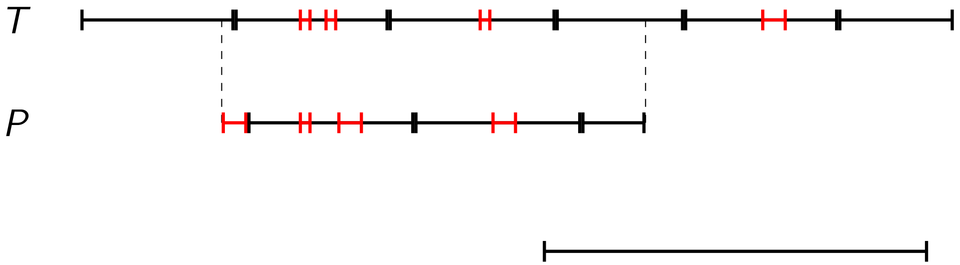

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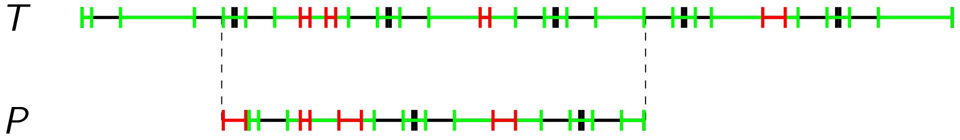

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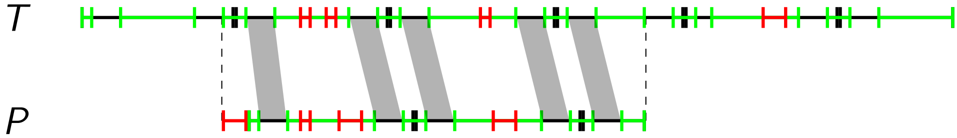

encoding cost $= \widetilde{\mathcal{O}}(k)$

Case $\boldsymbol{\Delta} = \widetilde{\mathcal{O}}(k)$ (small) $\Longrightarrow$ there exists alignment with the same cost of $\mathcal{X}$ that matches characters in the same uncovered black components.

Algorithmic Applications: Compressed Construction of $P^{\#}$ and $T^{\#}$

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- **Can we have fast construction of low-space representation of $P^{\#}, T^{\#}$, e.g. using grammars?**

Construction of $P^{\#}$ and $T^{\#}$

Theorem [KNW25]

Given S and $\mathrm{MI}(x)$ for all $x \in S$, we can construct a grammar-like representation $P^{\#}$ and $T^{\#}$ of size $\widetilde{\mathcal{O}}(k)$ in time $\widetilde{\mathcal{O}}(k^2)$. The grammar supports $\widetilde{\mathcal{O}}(1)$ time `PILLAR` operations.

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Theorem [KNW25]

Given N equality equations of the form $X[i..j) = X[i'..j')$ on a length-n string X, we can construct in time $\widetilde{\mathcal{O}}(N^2)$ a grammar-like representation of size $\widetilde{\mathcal{O}}(N)$ of a strings Y which:

1. satisfies all N equations, and
2. $Y[i] = Y[j]$ only when dictated by the equations.

Algorithmic Applications: Quantum Algorithms

Turning the Previous Ideas into an Algorithm

- Instead of computing directly $\mathrm{Occ}_k^H(P, T)$ compute in through $\mathrm{Occ}_k^H(P^\#, T^\#)$.

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- **Workaround:**
  - Find $\mathrm{Occ}_k^H(P, T) \subseteq C \subseteq \mathrm{Occ}_{5k}^H(P, T)$
  - Compute S s.t. $\{0, n-m\} \subseteq S \subseteq C$ and $\gcd(S) = \gcd(C)$

Constructing S through a Candidate Set

[CKW20]: Find a candidate set $\mathrm{Occ}_k^H(P, T) \subseteq C$ in one of two forms.

- $|C| = \mathcal{O}(k)$
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The set C satisfies $\mathrm{Occ}_k^H(P,T) \subseteq C \subseteq \mathrm{Occ}_{5k}^H(P,T)$.
Choose $\{0, n-m\} \subseteq S \subseteq C$ s.t. $\gcd(S) = \gcd(C)$ and $|S| = \mathcal{O}(\log n)$.

Construct compressed $P^{\#}$ and $T^{\#}$ and compute $\mathrm{Occ}_k^H(P^{\#}, T^{\#})$.

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➤ The number of queries are optimal for $k = o(n)$ (up to a logarithmic / subpolynomial factors).

Thanks!