On the Communication Complexity of Approximate Pattern Matching

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Pattern Matching (PM)

Text T, $|T| = n$ a b a a b a b a b a b b b a b b a a a b b b a b b a a

Pattern P, $|P| = m$ a b a b a b b b a b b a a

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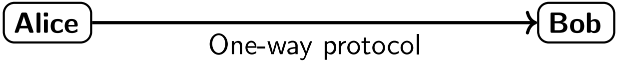
Communication Complexity



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$$\textbf{Alice} \xrightarrow[\text{One-way protocol}]{} \textbf{Bob}$$

1 Alice receives a PM instance.
$Text\ T$, Pattern P, Threshold k

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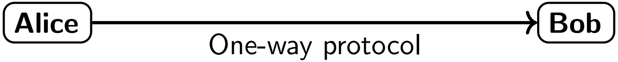

1. Alice receives a PM instance.
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Communication Complexity

Alice $\xrightarrow{\hspace{6em}}$ **Bob**

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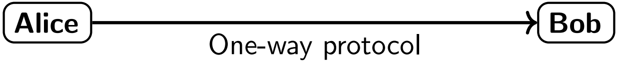
Communication Complexity



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Communication Complexity = "minimum # of machine words to send to Bob"

Example: Exact PM

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
Text T	a	b	b	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b	b
Pattern P	a	b	a	a	b	a	a	b	a	a	b	a	a	b	a	a	b													

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PM with mismatches	$\mathcal{O}(k)$	$\Omega(k)$	**[CKP19]**
PM with edits	$\mathcal{O}(k^3)$		**[CKW20]**
PM with edits	$\mathcal{O}(k \log m)$	$\Omega(k)$	**This Work**

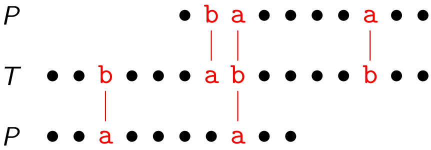
Finding a Period Structure

Suppose Alice takes a set S of alignments,

```
P           a b a b b a a a b b
            | | | | | | | | | |
T a b b b a a a b b b a a b b b
  | | |\ \ \ \ \| |
P a b a b b a a a b b
```

Finding a Period Structure

Suppose Alice takes a set S of alignments, and sends to Bob only the information about edits.



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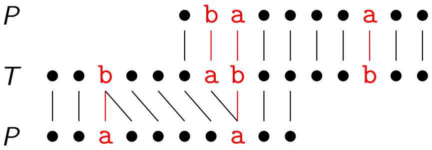

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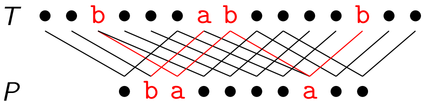

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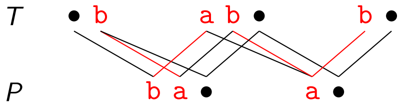

- Bob reconstructs the alignments in S.
- Bob makes a graph out of it.
- Bob selects red connected components,

.

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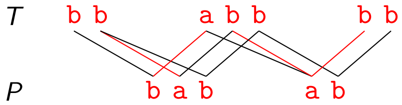

- Bob reconstructs the alignments in S.
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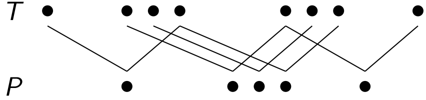

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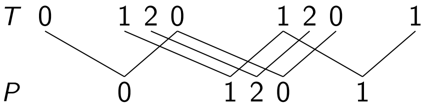

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- Bob selects black connected components, and numbers them.

Finding a Period Structure

Suppose Alice takes a set S of alignments, and sends to Bob only the information about edits.

T	0			1	2	0			1	2	0		1
P				0			1	2	0		1		

- Bob reconstructs the alignments in S.
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Finding a Period Structure

Suppose Alice takes a set S of alignments, and sends to Bob only the information about edits.

$T_{|S}$ 0 1 2 0 1 2 0 1

$P_{|S}$ 0 1 2 0 1

- Bob reconstructs the alignments in S.
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Mapping Back the Periodic Structure to the Original Strings

$T_{|S}$ **0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5**

$P_{|S}$ **0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5**

Mapping Back the Periodic Structure to the Original Strings

$T_{|S}$ **0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5**

T b **0 1 2 3 4 5 6 7 8** a **0 1 2 3** b **4 5 6 7 8 0 1 2 3** a a **4 5 6 7 8** a **0 1 2 3 4 5 6 7 8 0 1 2** a **3 4 5**

$P_{|S}$ **0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5**

P a a **0 1 2 3 4 5 6 7 8 0 1 2 3** b b **4 5 6 7 8 0 1 2 3** a **4 5**

Mapping Back the Periodic Structure to the Original Strings

$T_{|S}$ 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5

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τ_0 τ_1 τ_2 τ_3 τ_4

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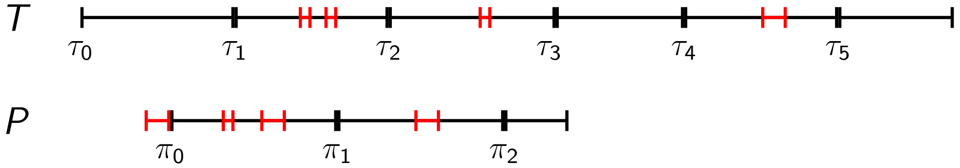
π_0 π_1 π_2

Alignments Covered by S



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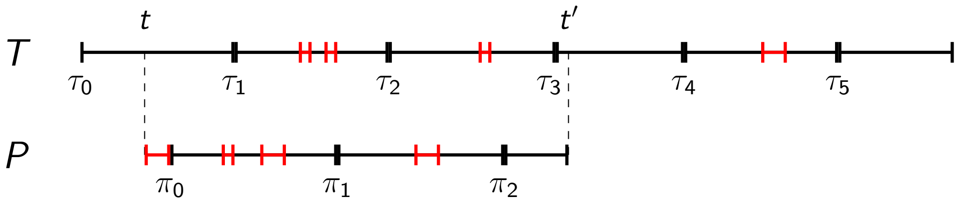

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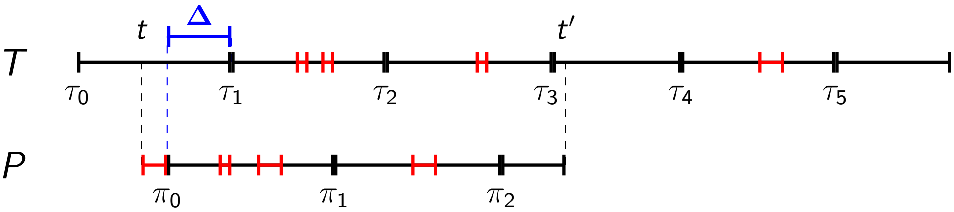

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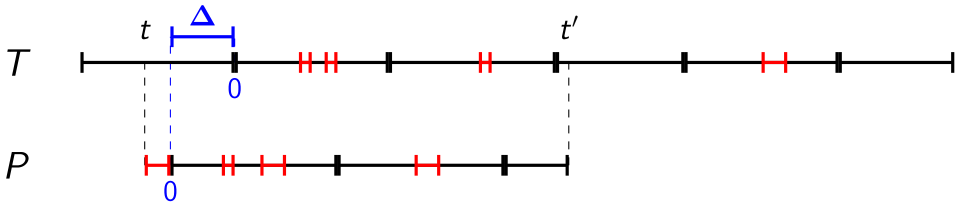

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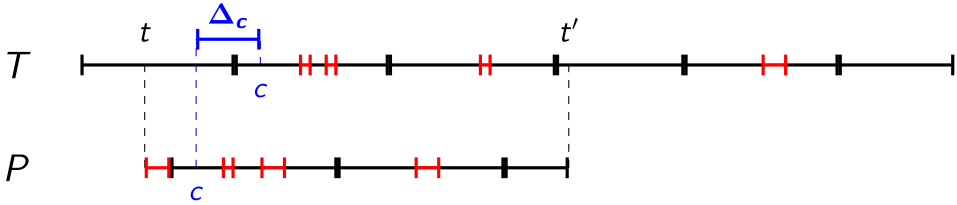

Case $\Delta = \widetilde{\Omega}(k)$ (large) $\implies$ if $\mathcal{X}$ is added to S, the # of black components at least halves.

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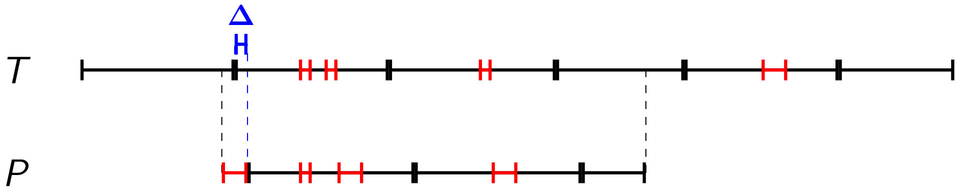

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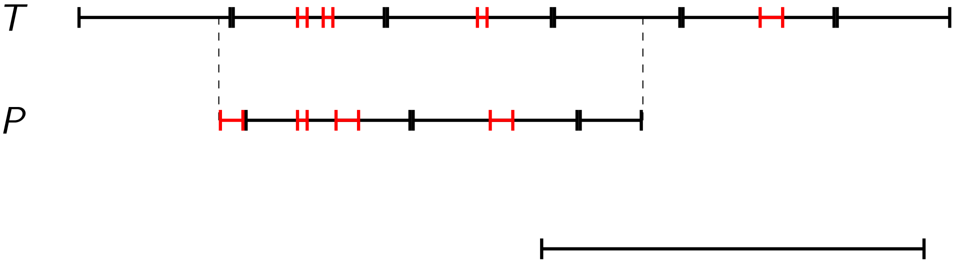

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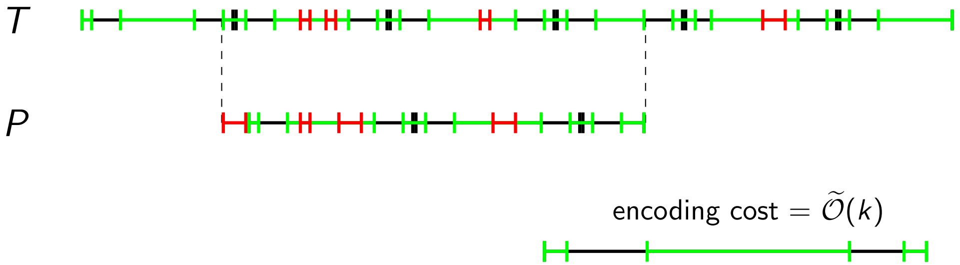

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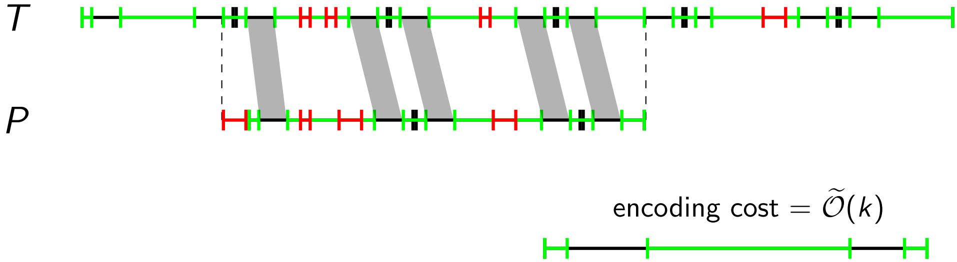

Case $\boldsymbol{\Delta} = \widetilde{\mathcal{O}}(k)$ (small) $\implies$ there exists alignment with the same cost of $\mathcal{X}$ that matches characters in the same uncovered black components.

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We apply our communication complexity results to the quantum setting:

- Input string S given as oracle where queries can be made in **superposition**

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Theorem [KN**W'24]**

There is a quantum algorithm that, given a PM with edits instance:

- computes $\mathrm{Occ}_k^E(P, T)$ using $\widehat{\mathcal{O}}(n/m \cdot \sqrt{km})$ queries and $\widehat{\mathcal{O}}(n/m \cdot (\sqrt{k}m + k^{3.5}))$ time;
- decides whether $\mathrm{Occ}_k^E(P, T) \neq \emptyset$ using $\widehat{\mathcal{O}}(\sqrt{n/m} \cdot \sqrt{km})$ queries and $\widehat{\mathcal{O}}(\sqrt{n/m} \cdot (\sqrt{k}m + k^{3.5}))$ time.

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➤ The number of queries is optimal for $k = o(m)$ (up to a subpolynomial factor).

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Thanks!