# Quantum Speed-ups for String Synchronizing Sets, Longest Common Substring, and $k$-mismatch Matching

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March 15, 2024

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- **Time complexity** $T(n)$: also counts the number of elementary gates

# Longest Common Substring (LCS) Problem

**Input:** two strings $S_1, S_2 \in \Sigma^n$
**Output:** maximum length $d$ such that $S_1[i\,..\,i+d-1] = S_2[j\,..\,j+d-1]$ for some $i, j$

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- Small-alphabet input (CKPR'21)

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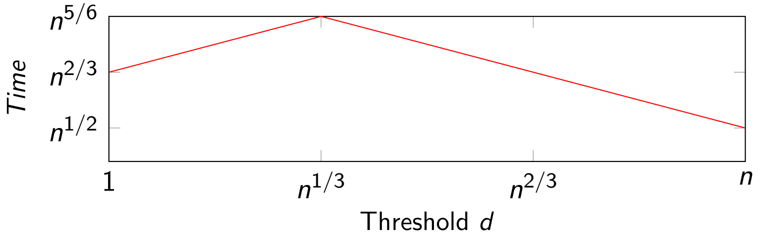


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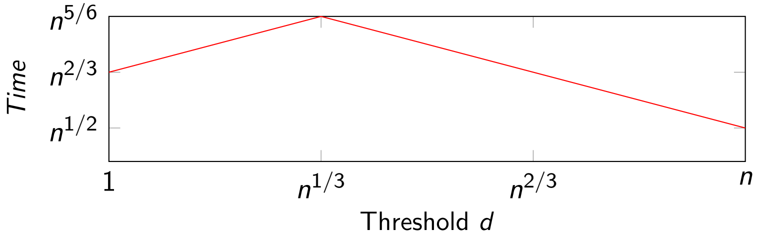


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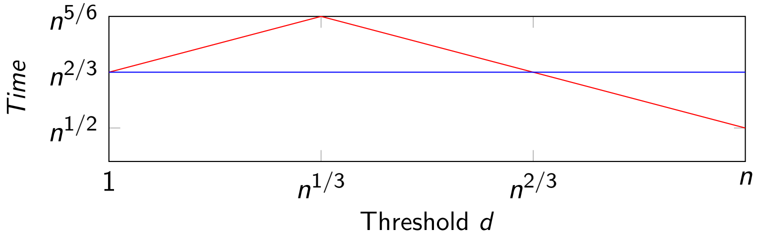


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By composing the two problems for the intermediate case $1 < d < n$ we obtain:

**Theorem (quantum query lower bound), based on [BHKKLS'11]**

Deciding LCS with threshold $d$ requires $\Omega(n^{2/3}/d^{1/6})$ quantum queries.

# Our results for LCS with threshold

**Theorem (an almost-tight upper bound) [Jin & N'23]**

Given $S_1, S_2 \in \Sigma^n$, there is a quantum algorithm that determines whether $\mathrm{LCS}(S_1, S_2) \geq d$ holds in

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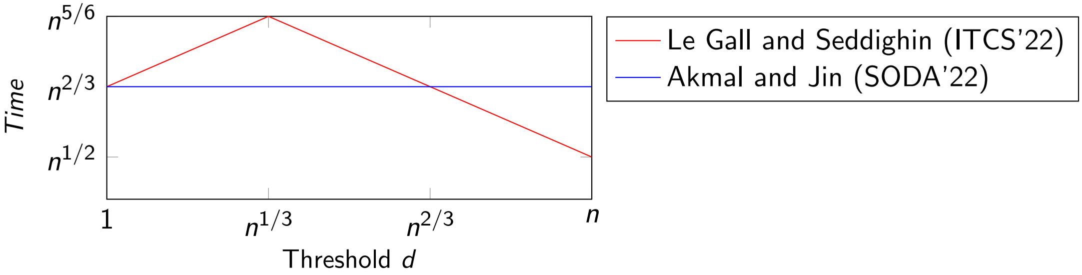

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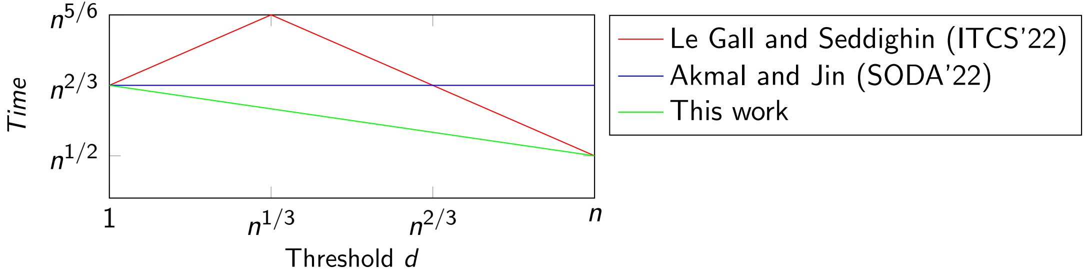

# LCS via Anchoring Technique

An *anchor set* C should satisfy: if $\mathrm{LCS}(S_1, S_2) \geq d$, then $S_1$ and $S_2$ must have a length-$d$ common substring anchored by C.

At least one common position must be included in C

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**Theorem [Akmal & Jin'22]**

Given a size-$m$ anchor set such that the $i$-th anchor can be reported in $\mathcal{T}$ quantum time, we can decide $\mathrm{LCS}(S_1, S_2) \geq d$ in $\widetilde{O}(m^{2/3} \cdot (\sqrt{d} + \mathcal{T}))$ quantum time.

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- Classical construction of synchronizing sets is slow.

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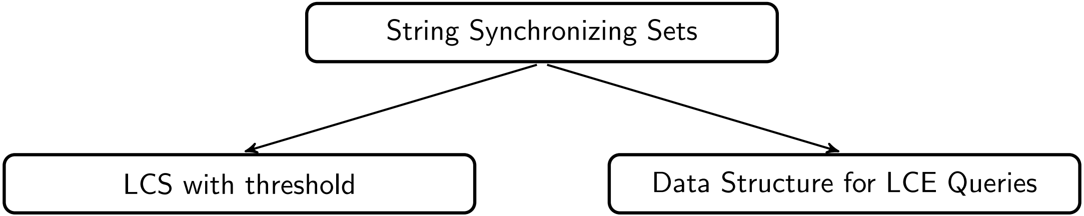
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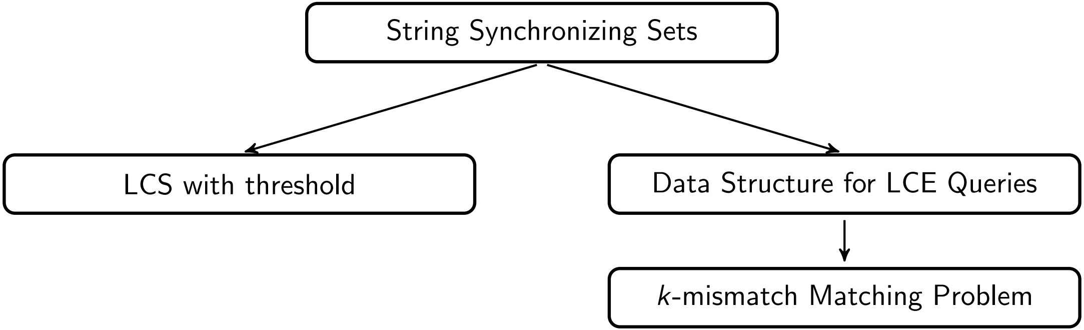
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# String Synchronizing Sets [Kempa & Kociumaka'19]

For a string $T[1\,.\,.\,n]$ and a positive integer $1 \le \tau \le n/2$, we say $\mathsf{A} \subseteq [1\,.\,.\,n-2\tau+1]$ is a *$\tau$-synchronizing set of $T$* if it satisfies the following properties:

Example: $\tau = 3$
red: positions in A

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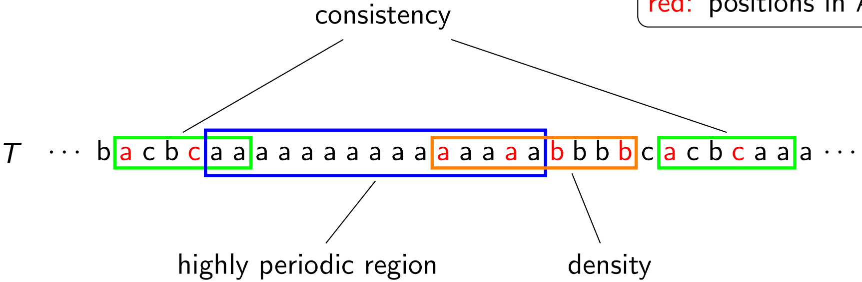

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Techniques: [KK'19] + Divide & Conquer + quantum minimum finding + [Vishkin'91]

Many recent applications in classical string algorithms:

- Sublinear-time Burrows-Wheeler Transform [KK'19]
- Optimal LCE data structure [KK'19]
- Longest Common Substring [CKPR'21]
- Dynamic & Compressed suffix arrays [KK'22], [KK'23]

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$\mathcal{T}_{\text{prep}} = 0$, $\mathcal{T}_{\text{ans}} = \widetilde{O}(\sqrt{n})$ ——— $\mathcal{T}_{\text{prep}} = O(n)$, $\mathcal{T}_{\text{ans}} = O(1)$

Tradeoff: $(\mathcal{T}_{\text{prep}} + \sqrt{n}) \cdot (\mathcal{T}_{\text{ans}} + 1) \ge \widetilde{\Omega}(n)$

# The (Hamming) $k$-mismatch Matching Problem

**Input:** a text $T \in \Sigma^n$, a pattern $P \in \Sigma^m$, and a threshold $k \in [m]$
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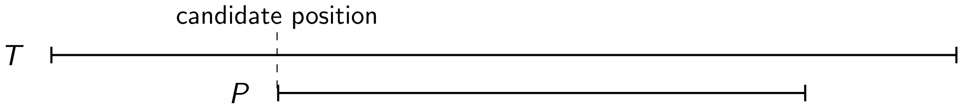
Their constructive algorithm for the theorem can be adapted to a quantum algorithm requiring $\widetilde{O}(\sqrt{km})$ quantum time and query complexity.

# Handling the two Cases

**Case (i):** $O(k)$ candidate positions for $k$-mismatch occurrences.



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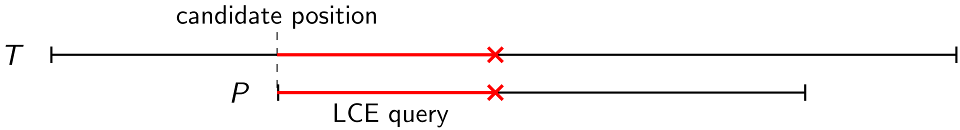

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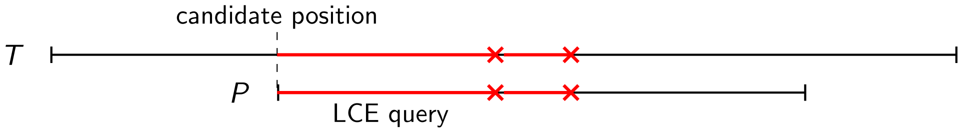

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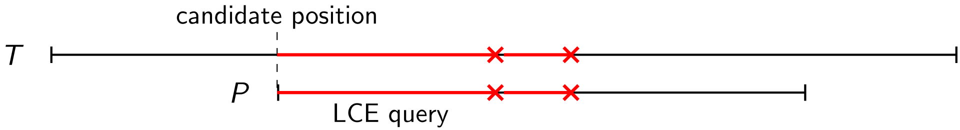


Speed up candidate position verification through LCE data structure:

$$\widetilde{O}\big(\underbrace{m/\tau^{\frac{1}{2}-o(1)}}_{\text{Preprocessing}} + \underbrace{\sqrt{k}\cdot k\sqrt{\tau}}_{\text{Grover's search } \times\ k \text{ LCE queries}}\big)$$

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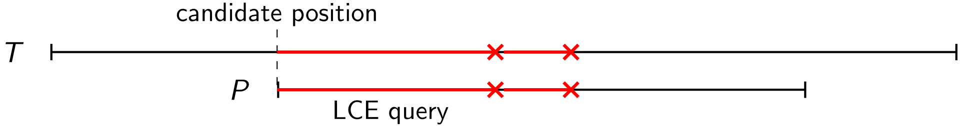


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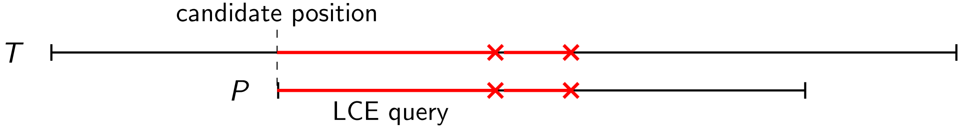


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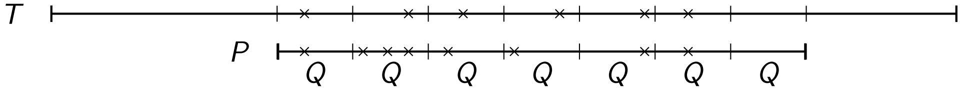

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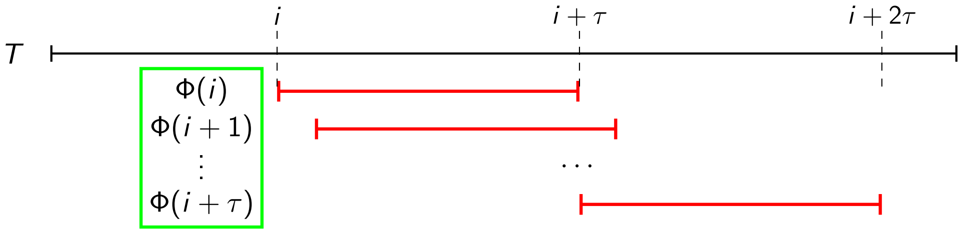
Thank you!

# Construction of $\tau$-synchronizing set A

We focus on 'non-periodic case': $\mathrm{per}(T[i \,..\, i+\tau]) > \tau/3$ for all $i$

Follow [KK'19]'s framework of 'picking local minimizers' of a hash function $\phi$:

- Choose $\phi : \Sigma^\tau \to \mathbb{Z}$
- Denote $\Phi(i) = \phi(T[i \,..\, i+\tau-1])$




$i \in$ A iff minimum is achieved at $\Phi(i)$ or $\Phi(i+\tau)$

Consistency and Density always hold.

# Construction of $\tau$-synchronizing set A

How to ensure **sparsity**?

- The hash function $\phi$ should guarantee probability $O(1/\tau)$ of $i$ being included in A.

How to make $\phi$ **efficiently computable** using few quantum queries?

- We need only ability to distinguish substrings with $\Omega(\tau)$ overlap: an adaptation of Deterministic Sampling [Vishkin'91] gives us this guarantee.
- To further speed up computability we structure $\phi$ such that one can find the minimal hash value in a tournament tree-like fashion.

By using quantum minimum finding for each level, we obtain the recursion

$$T(\tau) = \sqrt{b} \cdot (T(\tau/b) + \widetilde{O}(\sqrt{\tau})) \cdot O(\log \tau)$$
$$= \tau^{1/2+o(1)}$$