# Faster Tree Edit Distance via APSP Equivalence

Master Thesis Presentation

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Supervised by

**Adam Polak** (Bocconi University) **David Steurer**

with contributions from

**Barna Saha** (UC San Diego) **Virginia Vassilevska Williams** (MIT)
**Yinzhan Xu** (UC San Diego) **Christopher Ye** (UC San Diego)

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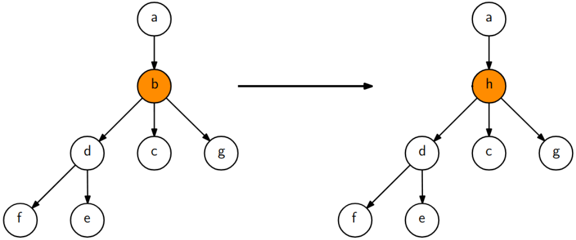

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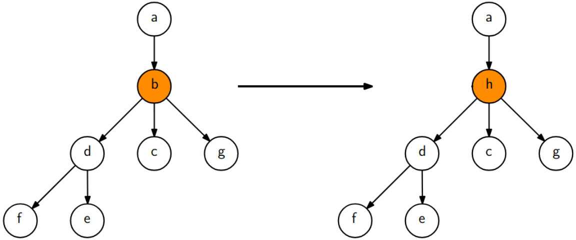


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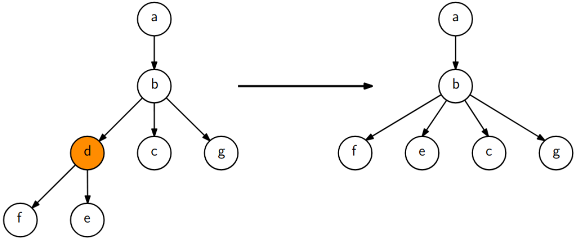

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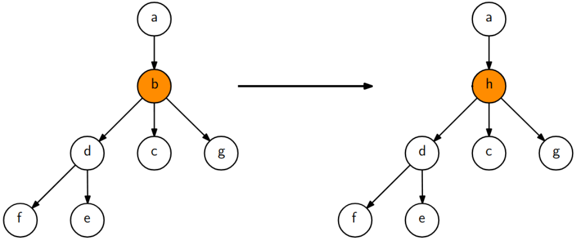


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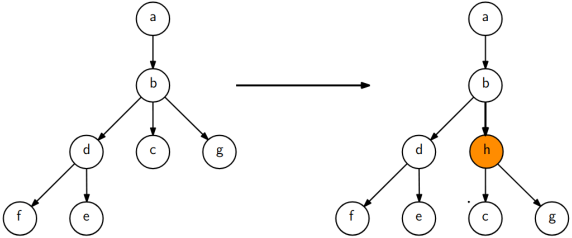


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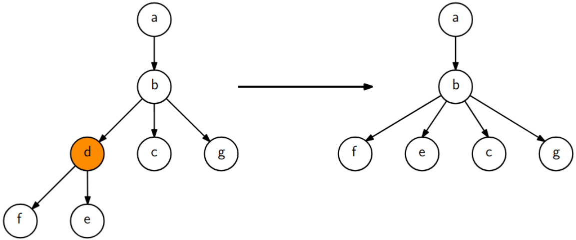


Images from Seddighin, Seddighin 2019

# TED reformulated

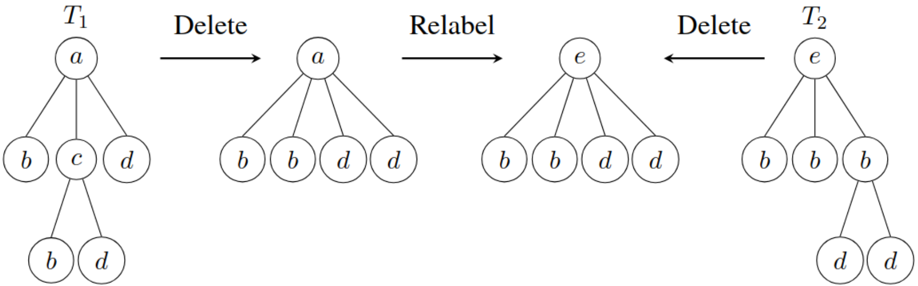


Image from Mao 2022

- Relabel $v$ to $v'$ with cost $\delta(v, v')$.
- Delete $v$ from $T_1$ with cost $\delta(v, \varepsilon)$.
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Last three fall within decomposition strategy framework formalized in [Dulucq and Touzet, 2003].
For algorithms within the framework a $\Omega(n^3)$ lower bound exists.

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$$\begin{tikzcd}
\text{All-Pair Shortest Paths} \arrow[d, bend right] \\
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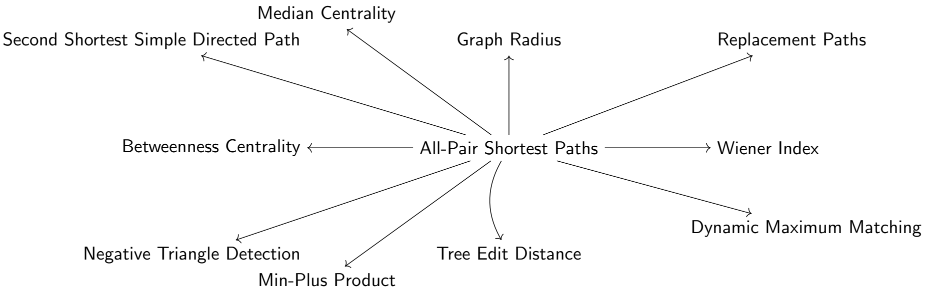

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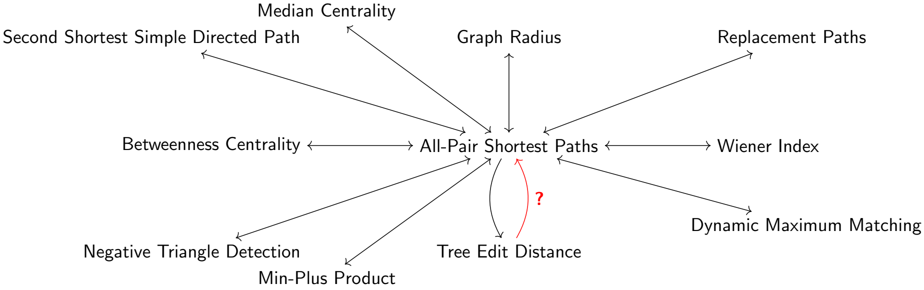

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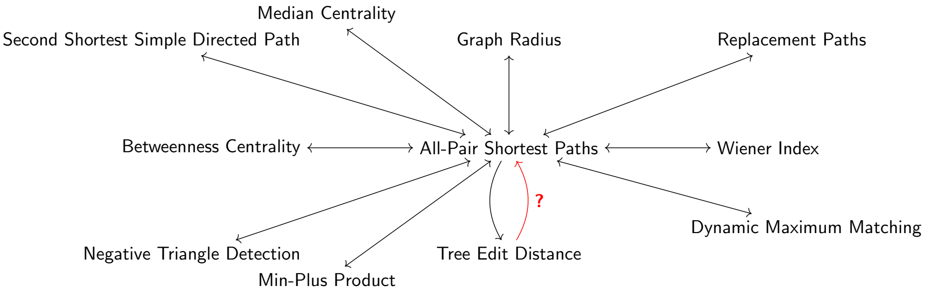
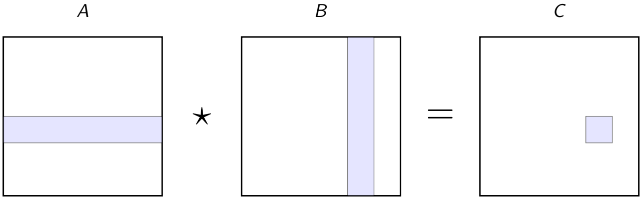



**Question 2: is TED equivalent to APSP?**

# Unweighted Tree Edit Distance

Key component to achieve truly subcubic algorithms for unweighted TED:

**Monotone Min-plus Product**




$$C_{i,j} = \min_{1 \le k \le n}\{A_{i,k} + B_{k,j}\}$$

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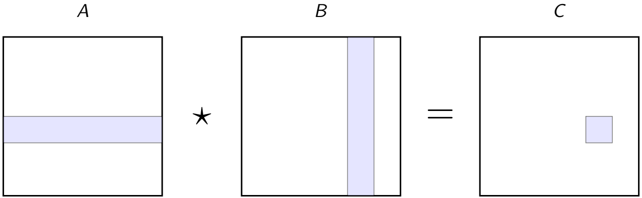


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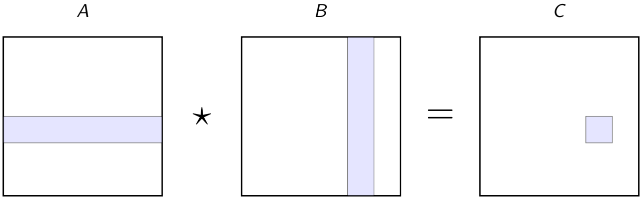


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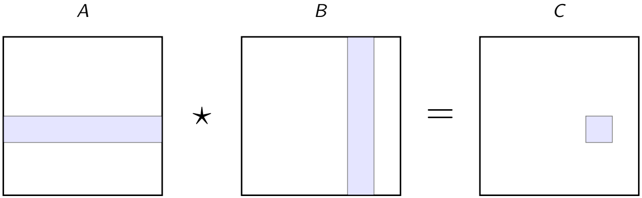


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# Sketch of the Reduction

# Similarity of Strings

Instead of computing the edit distance between two strings $A = a_1 \cdots a_n$, $B = b_1 \cdots b_n$, we compute the similarity between $A, B$.

`m i t a g s e m i n a r`

`c h r i s t m a s`

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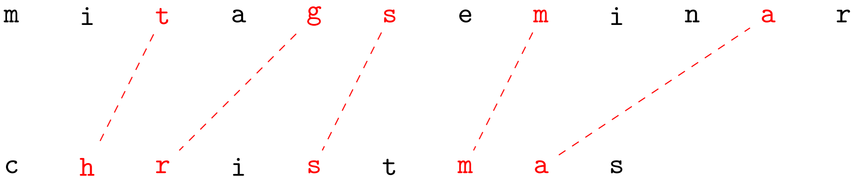


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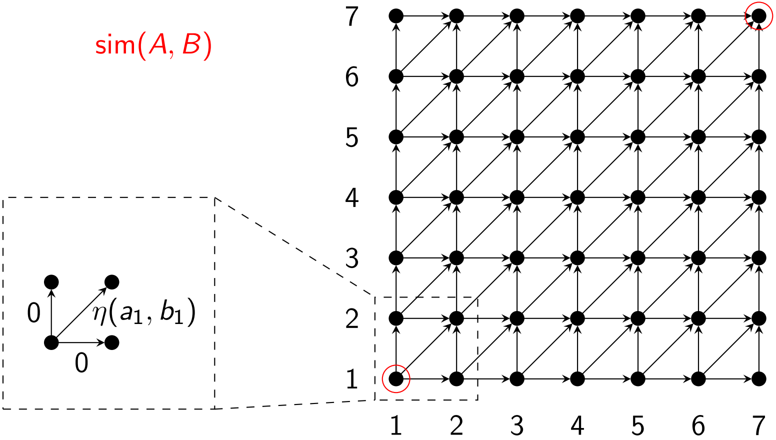

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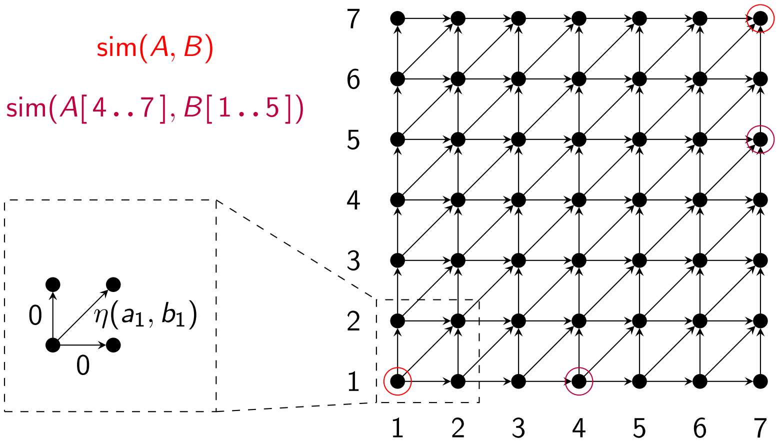

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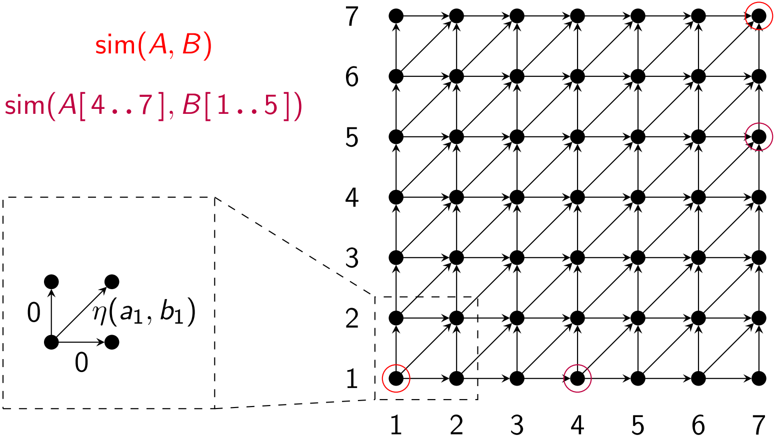


**lower-to-right border:**

suffix vs prefix

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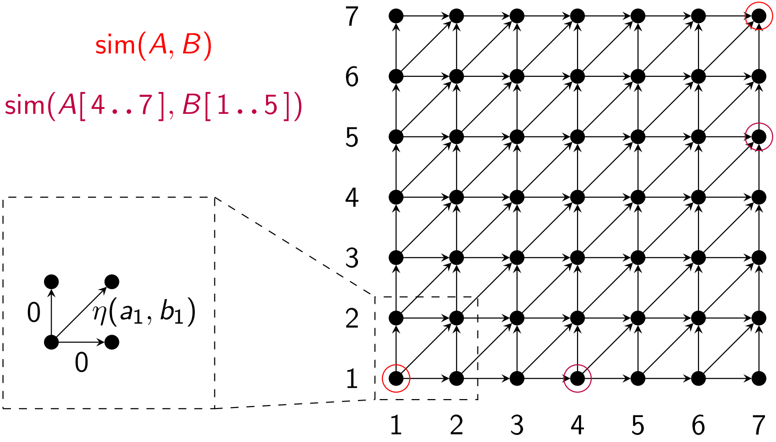


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# String Alignment Graphs

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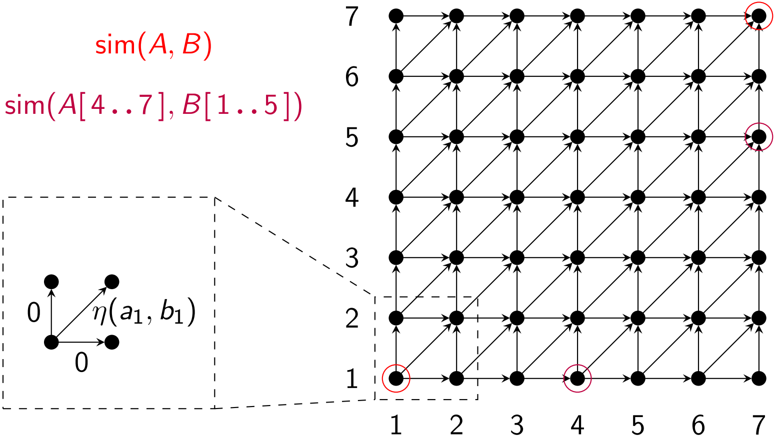



**lower-to-right border:**
suffix vs prefix

**lower-to-upper border:**
infix vs whole

**left-to-right border:**
whole vs infix

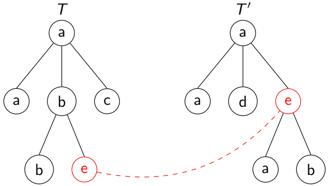
**lower-to-right border:**
prefix vs suffix

$\mathcal{O}(n^2)$ to compute all

**Bedtime reading:** "Semi-local string comparison: algorithmic techniques and applications" by Alexander Tiskin

# Similarity of Trees

We compute the similarity between $T$ and $T'$.




$\eta(v, v') \coloneqq \delta(v, \varepsilon) + \delta(\varepsilon, v') - \delta(v, v')$ "how much I save by substituting $v$ with $v'$"

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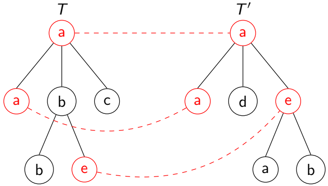


$\eta(v, v') := \delta(v, \varepsilon) + \delta(\varepsilon, v') - \delta(v, v')$ "how much I save by substituting $v$ with $v'$"

$\mathrm{sim}(T, T') =$ "maximum weight of similarity matching"

Condition on similarity matching: for any two matched vertices $(v, v')$ and $(u, u')$

- $v$ is an ancestor of $u$ in $T$ if and only if $v'$ is an ancestor of $u'$ in $T'$,
- $v$ comes before $u$ in the pre-order of $T$ if and only if $v'$ comes before $u'$ in the pre-order of $T'$.

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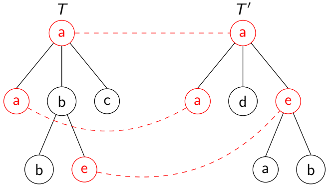


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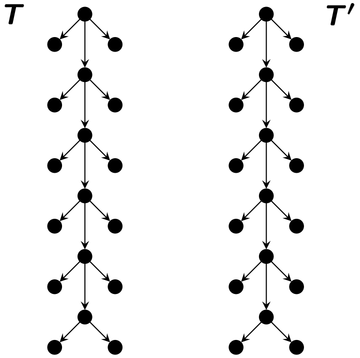
$\mathrm{sim}(T, T') = \sum_{v \in T} \delta(v, \varepsilon) + \sum_{v' \in T'} \delta(\varepsilon, v') - \mathrm{ted}(T, T')$.

# TED on Caterpillar Trees I

Let us start by computing the TED between two caterpillar trees...



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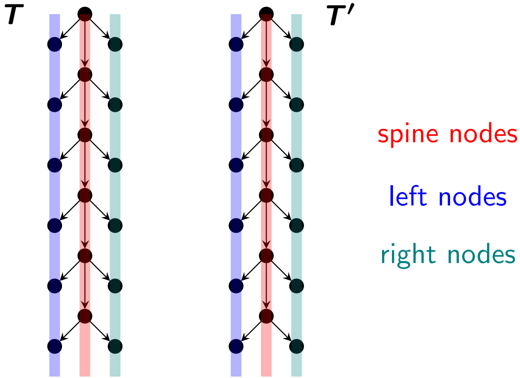

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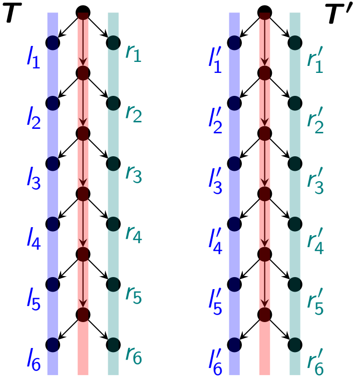
$L = l_1 l_2 l_3 l_4 l_5 l_6$

$R = r_1 r_2 r_3 r_4 r_5 r_6$

$L' = l'_1 l'_2 l'_3 l'_4 l'_5 l'_6$

$R' = r'_1 r'_2 r'_3 r'_4 r'_5 r'_6$




spine nodes

left nodes

right nodes

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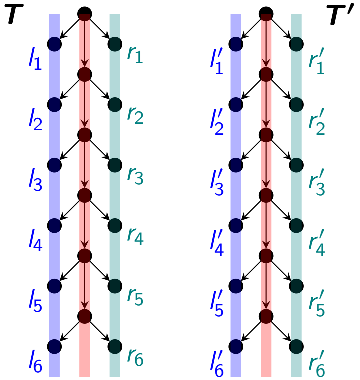


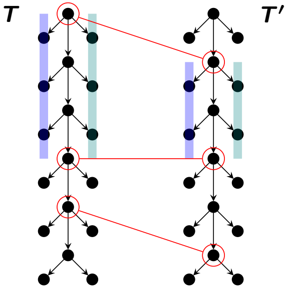

spine nodes

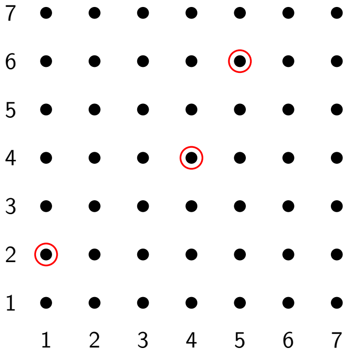
left nodes

right nodes

...with the assumption that spine, left and right nodes of $T$ only match with nodes of their same type (color) in $T'$, respectively.

# TED on Caterpillar Trees II



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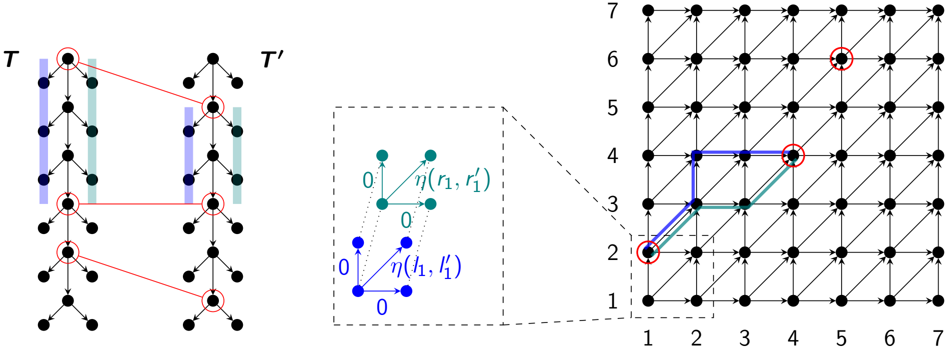

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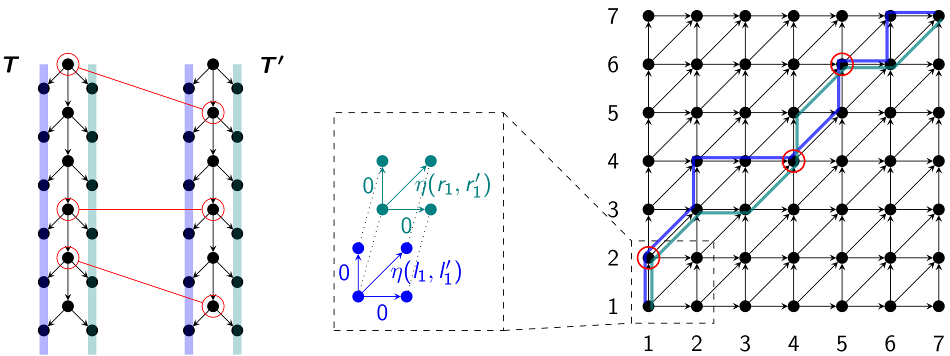
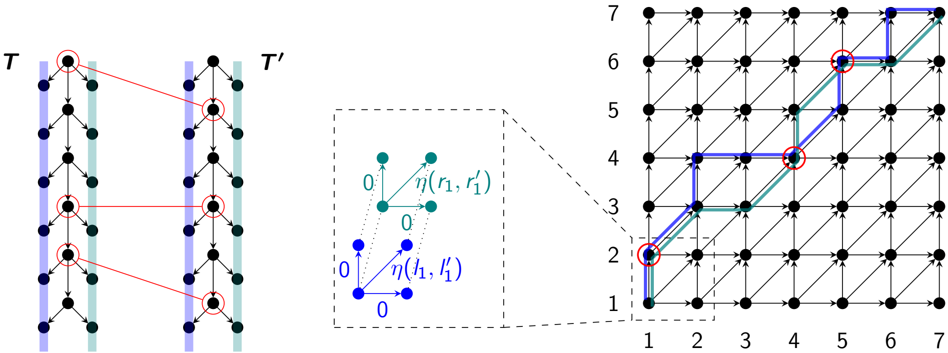
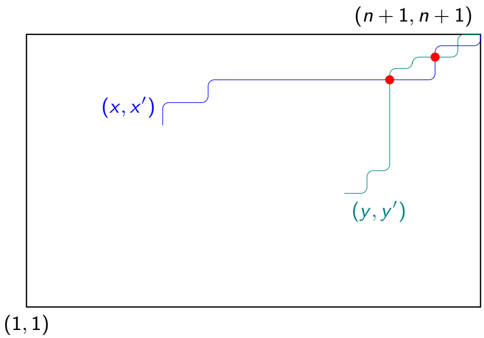





sim($T$, $T'$) equals to the maximum achievable sum of:

1. the weight of a path from $(1,1)$ to $(n+1, n+1)$ in the alignment graph of $\mathrm{sim}(L, L')$;
2. the weight of a path from $(1,1)$ to $(n+1, n+1)$ in the alignment graph of $\mathrm{sim}(R, R')$; and
3. values $\eta(c_i, c'_{i'})$ for $(i, i')$ where the two paths intersect (each $c_i$ and $c'_{i'}$ appears at most once).

# A Divide and Conquer Scheme for TED on Caterpillar Trees I




sim$((x, x'), (y, y'))$ equals to the maximum achievable sum of:

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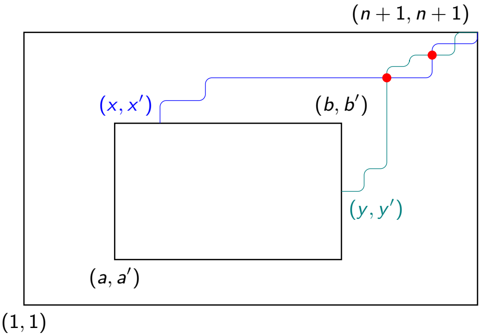


**Divide et Conquer Scheme**

**Input:**

- The lower-left corner $(a, a')$ and upper-right corner $(b, b')$ of a rectangle.

**Output:**

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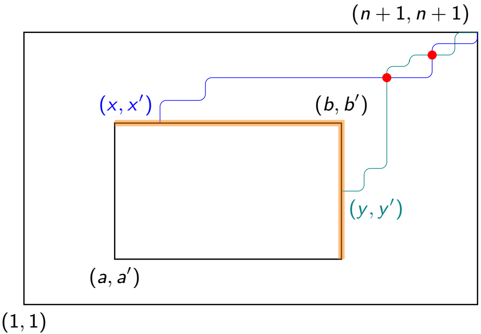


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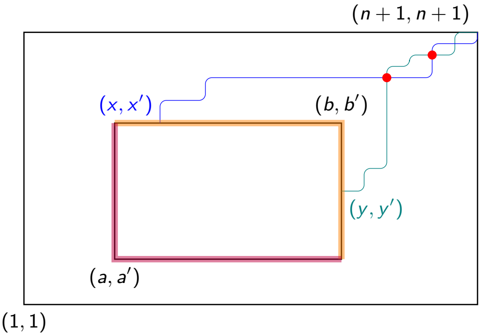


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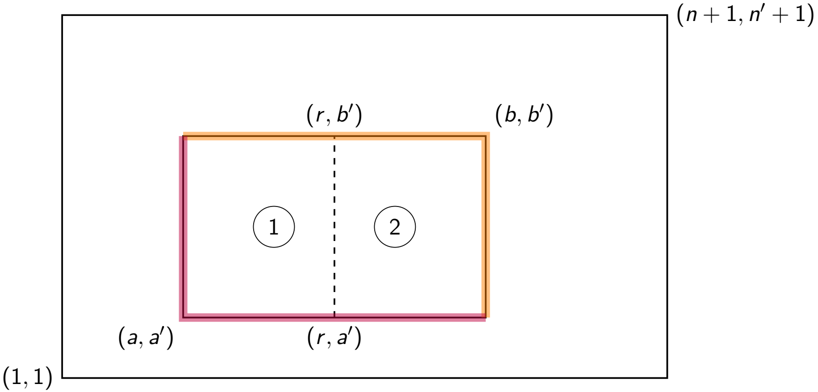

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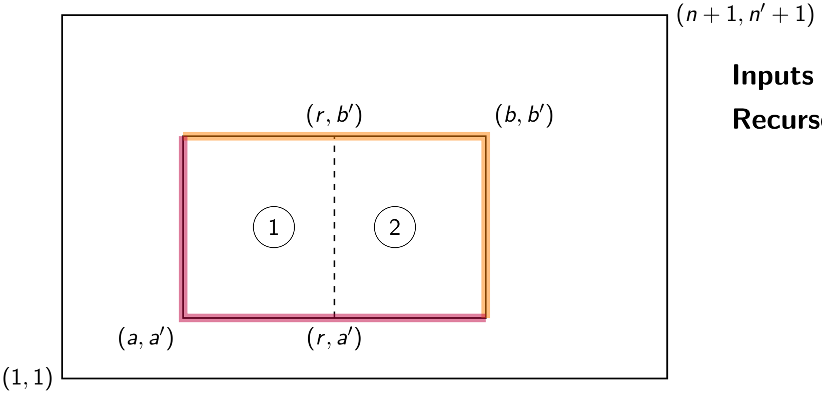


**Inputs of subrectangle 2.** ✔

**Recurse on subrectangle 2.** ✔

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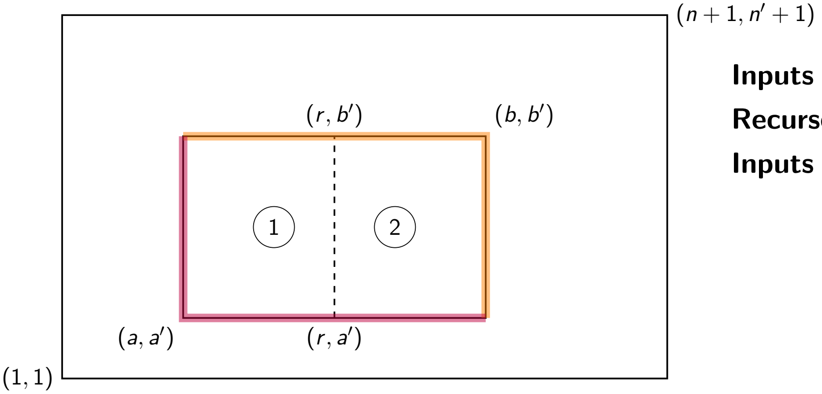


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**Inputs of subrectangle 1.**

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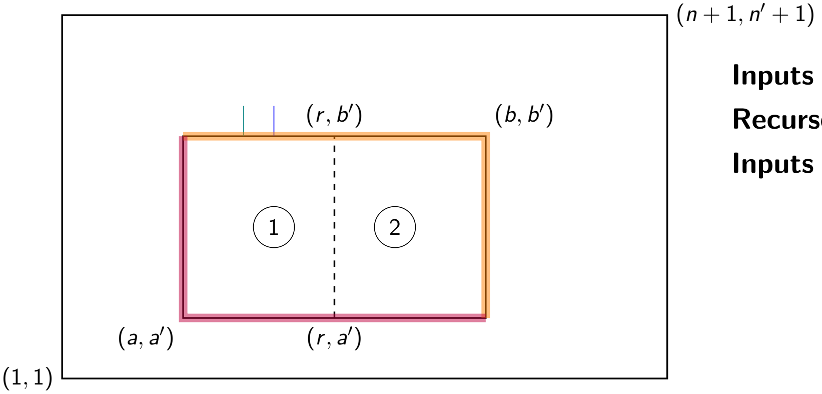


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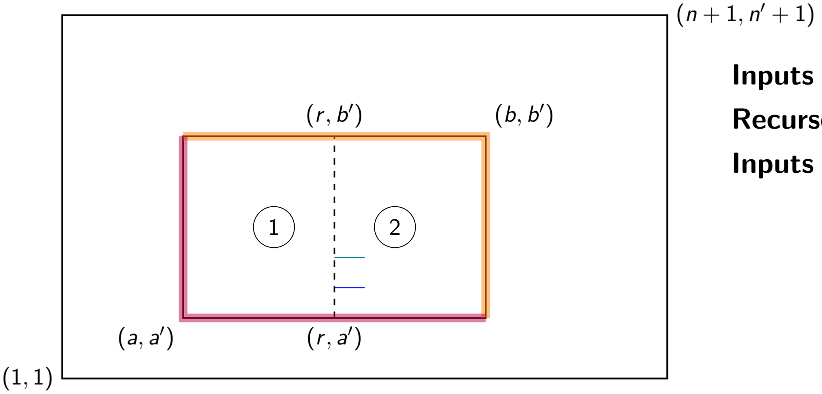


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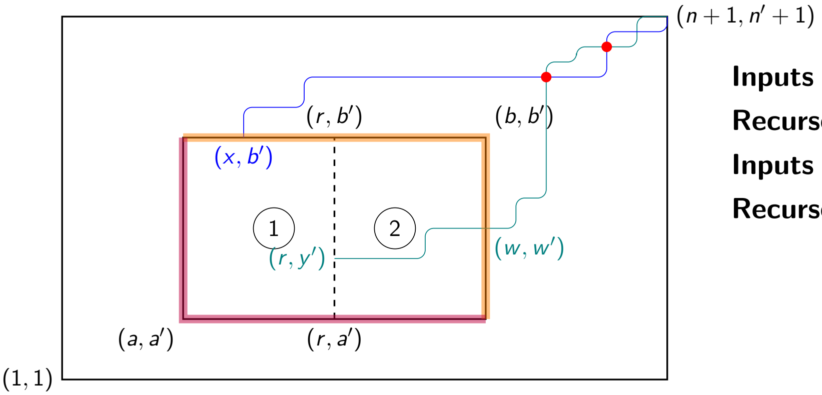


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**Inputs of subrectangle 1.** ✔

**Recurse on subrectangle 1.** ✔

$$\mathrm{sim}((x,b'),(r,y')) = \max_{(w,w')} \left\{ \mathrm{sim}(R[r\,..\,w], R'[y'\,..\,w']) + \mathrm{sim}((x,b'),(w,w')) \right\}.$$

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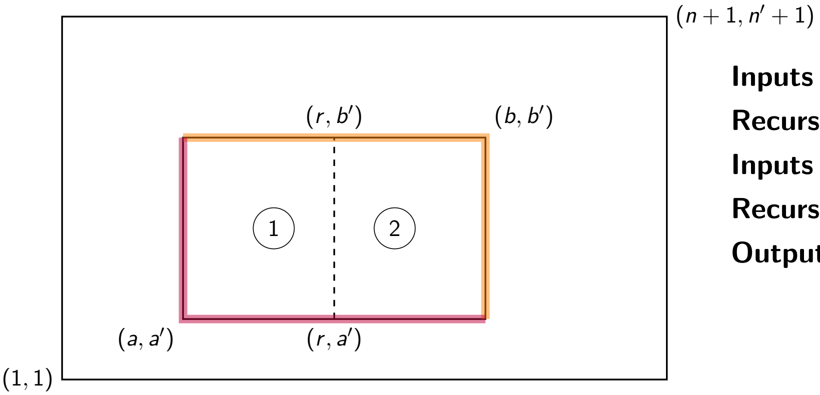



**Inputs of subrectangle 2.** ✔

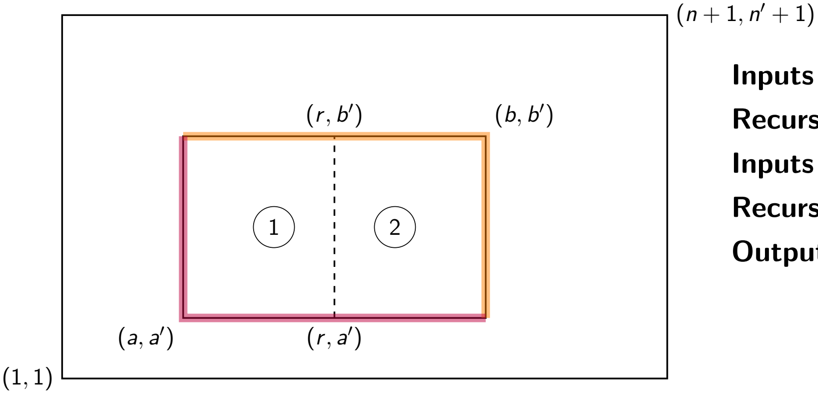
**Recurse on subrectangle 2.** ✔

**Inputs of subrectangle 1.** ✔

**Recurse on subrectangle 1.** ✔

**Outputs of big rectangle.**

# A Divide and Conquer Scheme for TED on Caterpillar Trees II

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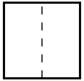
# A Divide and Conquer Scheme for TED on Caterpillar Trees III

Since we can handle vertical “cuts”, we can also handle horizontal “cuts”.



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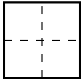

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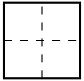

We obtain the recurrence

$$T(n) = 4T(n/2) + \mathcal{O}(T_{\text{APSP}}).$$

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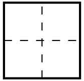

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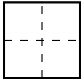

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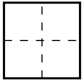

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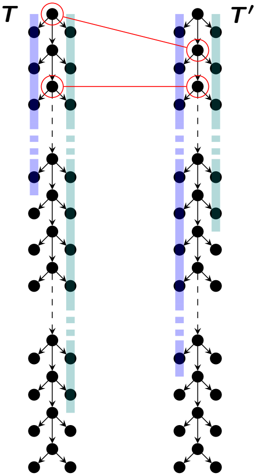
**Q: How to drop the assumption that spines, left and right nodes map only nodes of the same type?**

# TED on Caterpillar Trees: Dropping the Assumption



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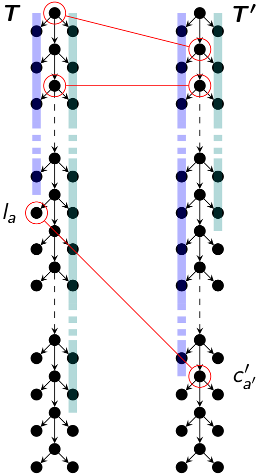

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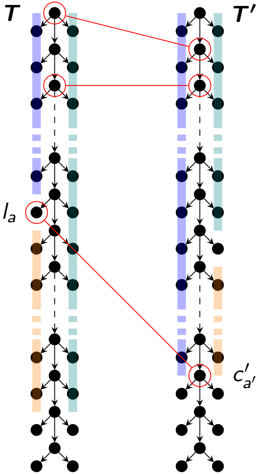

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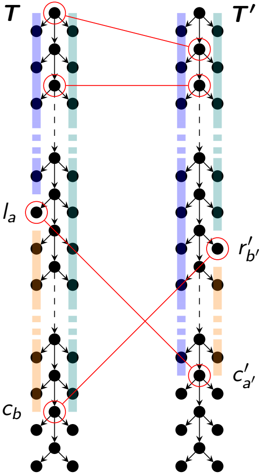

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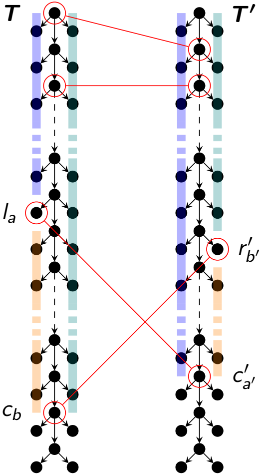


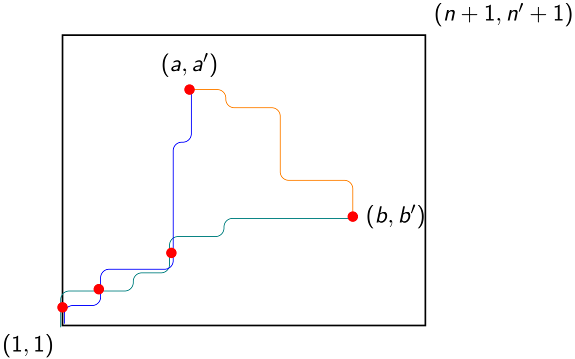

# From Caterpillar Trees to Arbitrary Trees

We generalize TED on caterpillar trees to Spine Edit Distance.

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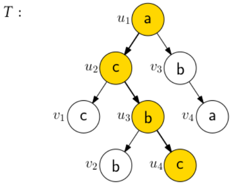
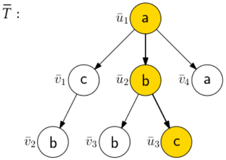
**Input:** trees $T$, $T'$, root-to-leaf paths $S \subseteq T$, $S' \subseteq T'$, and $\mathrm{sim}(\mathrm{sub}(v), \mathrm{sub}(v'))$ $\forall (v, v') \in (T \times T') \setminus (S \times S')$.



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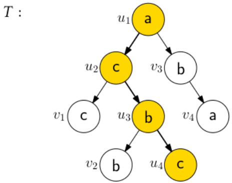


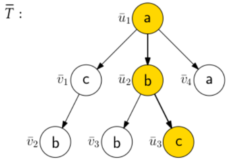


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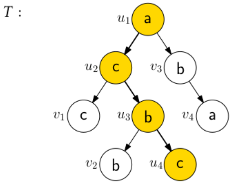
**Why?**

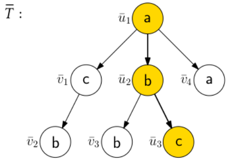
- There are still spine, left, and right nodes.
- The underlying paths are not in string alignment graphs anymore but in *forest alignment graphs*.

# From Caterpillar Trees to Arbitrary Trees

We generalize TED on caterpillar trees to Spine Edit Distance.

**Input:** trees $T$, $T'$, root-to-leaf paths $S \subseteq T$, $S' \subseteq T'$, and $\mathrm{sim}(\mathrm{sub}(v), \mathrm{sub}(v'))$ $\forall (v, v') \in (T \times T') \setminus (S \times S')$.




**Output:** $\mathrm{sim}(\mathrm{sub}(v), \mathrm{sub}(v'))$ $\forall (v, v') \in S \times S'$.

**Why?**

- There are still spine, left, and right nodes.
- The underlying paths are not in string alignment graphs anymore but in *forest alignment graphs*.

We also prove that TED on arbitrary trees is fine-grained equivalent to Spine Edit Distance.

# Thanks!