Hardness of Tree Edit Distance and Friends

Bingbing Hu[1] **Jakob Nogler**[2] Barna Saha[1]

[1]UC San Diego

[2]MIT

(String) Edit Distance

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"Unweighted" (String) Edit Distance: all costs are one

(String) Edit Distance Algorithms

References	Time	Remarks
Vin68, NW70, Sel74, WF74	$\mathcal{O}(n^2)$	textbook algorithm
MP80	$\mathcal{O}(n^2/\log^2 n)$	small alphabets and integer weights only
BF05	$\mathcal{O}(n^2 \log\log n/\log^2 n)$	small integer weights only
BI18	$\Omega(n^{2-o(1)})$	under SETH, already for unweighted

Dynamic (String) Edit Distance

Strings undergo updates (insertion/deletions and substitutions), and we need to maintain their edit distance.

$$\text{ed}(\ \texttt{port}\ ,\ \texttt{arts}\ ) = 3$$

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References	**Update Time**	**Remarks**
CKM20	$\widetilde{\mathcal{O}}(n)$	unweighted
CKM20	$\widetilde{\mathcal{O}}(n \cdot \min(\sqrt{n}, W))$	uniform weights, W is maximum weight
GK25	$\widetilde{\mathcal{O}}(nW)$	arbitrary weights
CKW23	$\Omega(n^{1.5-o(1)})$	under APSP, large W

This Paper: Generalization of (String) Edit Distance in the Dynamic Setting

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$\Downarrow$ Generalization on Trees

Tree Edit Distance Problem (TED)
Input: Two rooted, labeled, left-to-right-ordered trees T_1, T_2 and a cost function δ.
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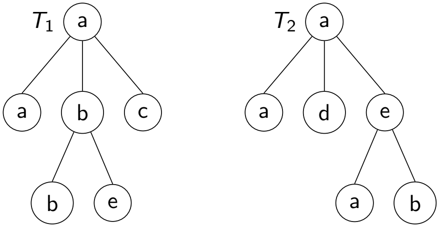

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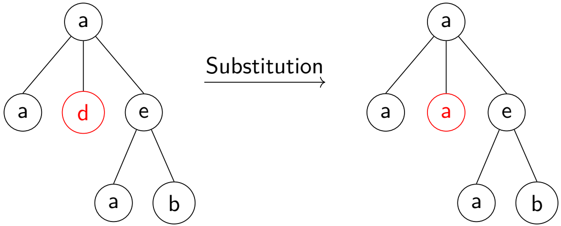

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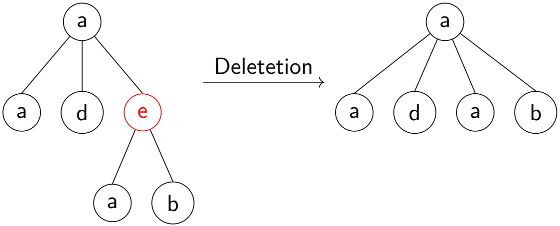

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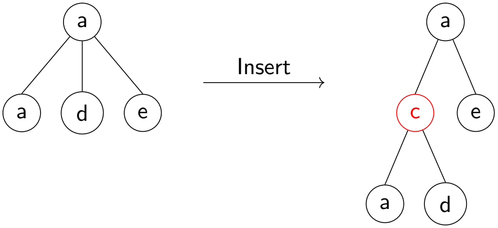

Algorithms for Tree Edit Distance

Reference	Complexity	Remarks
Tai79	$\mathcal{O}(n^6)$	weighted
SZ89	$\mathcal{O}(n^4)$	weighted
Klein98	$\mathcal{O}(n^3 \log n)$	weighted
DMRW10	$\mathcal{O}(n^3)$	weighted
BGMW20	no $\mathcal{O}(n^{3-\varepsilon})$ algo under APSP	weighted
NPSVWXY25	$n^3/2^{\Omega(\sqrt{\log n})}$	weighted
Mao22	$\mathcal{O}(n^{2.9546})$	unweighted
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Question 2: Is Unweighted Edit Distance on trees harder than on strings?

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$\Downarrow$ About Balanced Sequence of Parenthesis

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Input: A string S over a set of parenthesis
Output: Closest S' to S w.r.t. edit distance such that S' is balanced

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$$S = \textbf{((\{)]} \quad \longrightarrow \quad S' = \textbf{(\{ \})}$$

distance $= 3$

RNA Folding

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Input: A string S over alphabets $\Sigma \cup \Sigma'$, where each $\sigma \in \Sigma$ has a matching symbol $\sigma' \in \Sigma'$.
Output: Max. number of intersection-free connections between matching symbols σ, σ'.

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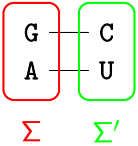

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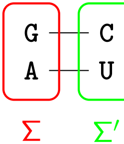

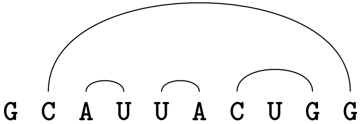

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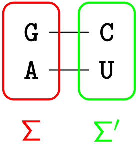

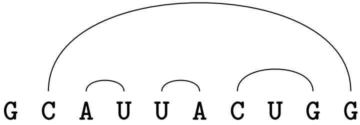

Relation with edit distance: we can embed edit distance in RNA folding.

Algorithms for Dyck Edit Distance and RNA Folding

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Our Results: Bad News...

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Theorem

For any $\varepsilon > 0$, there are no dynamic algorithms:

- for weighted TED with $\mathcal{O}(n^{3-\varepsilon})$ update time, unless Weighted 4-Clique Conj. fails;
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Thanks!