The Communication Complexity of Pattern Matching with Edits Revisited

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Pattern Matching (PM)

Text T, $|T| = n$ a b a a b a b a b a b b b a b b a a a b b b a b b a a

Pattern P, $|P| = m$ a b a b a b b b a b b a a

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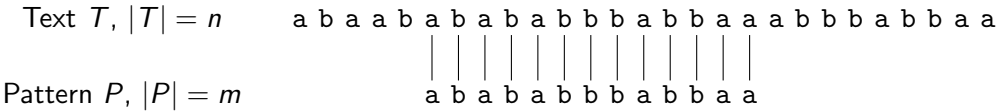

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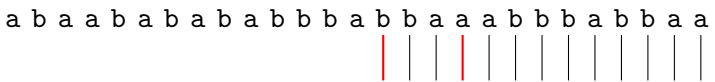

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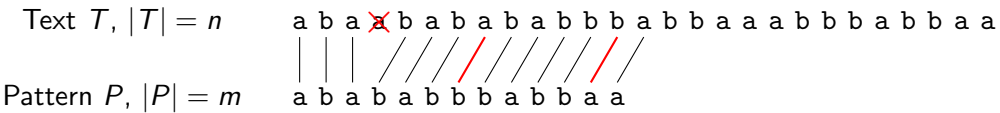

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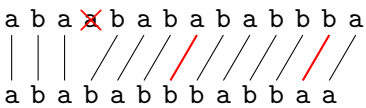

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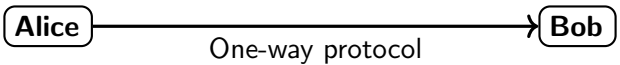

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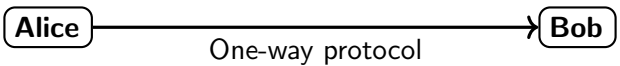

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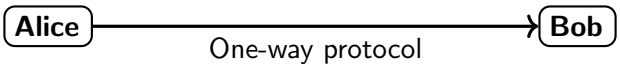

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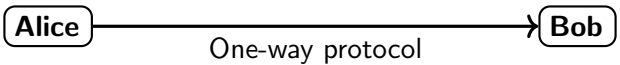

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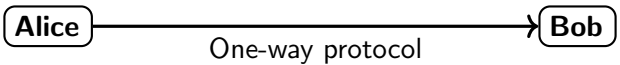

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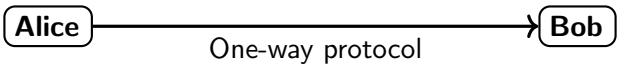

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Communication Complexity = "minimum bits to send to Bob"

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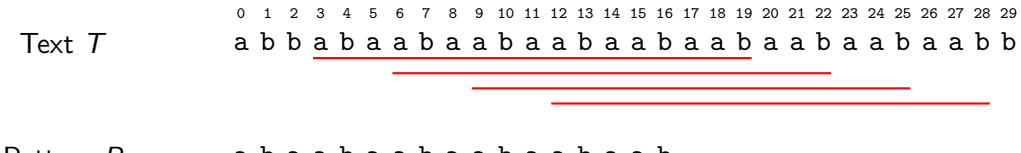

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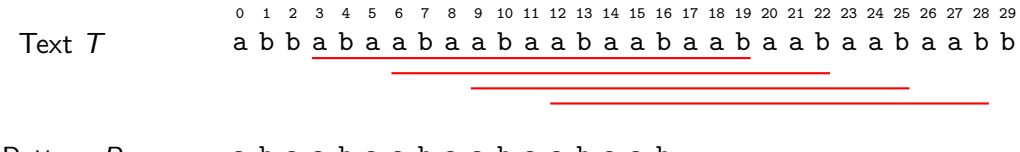

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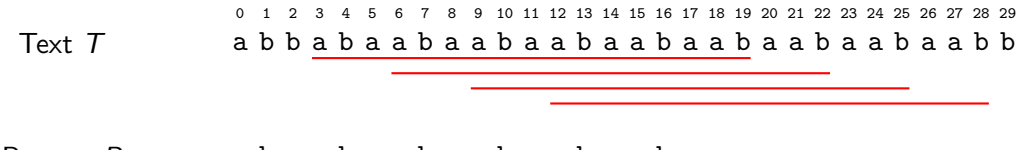

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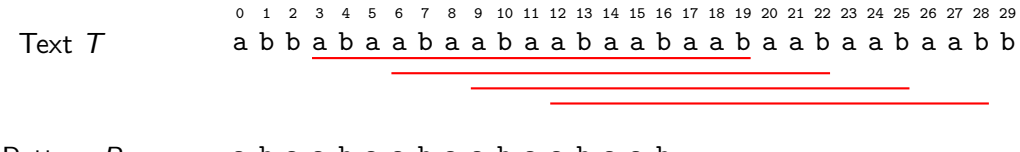

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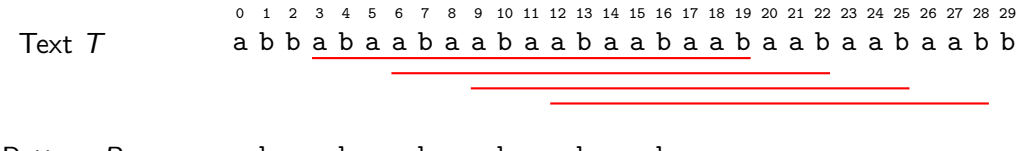

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3. She can send P and T directly.

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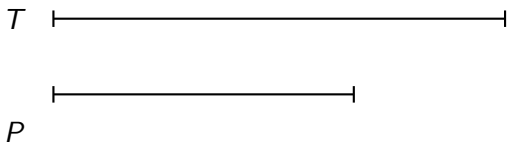

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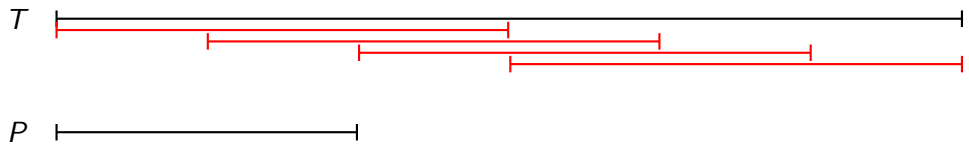

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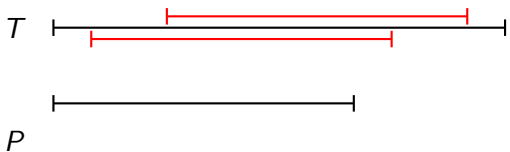

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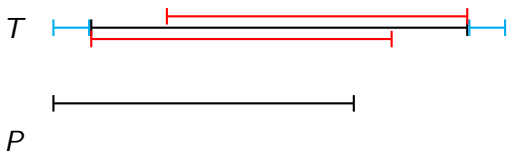

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- The last results are at the basis of our quantum algorithms [K**N**W24], [K**N**W25].

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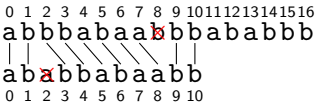

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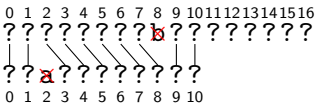

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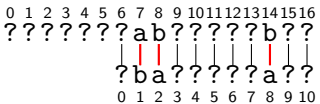

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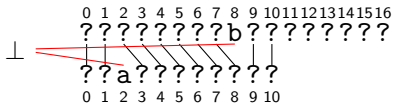

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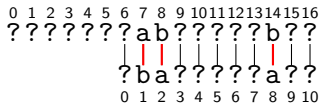

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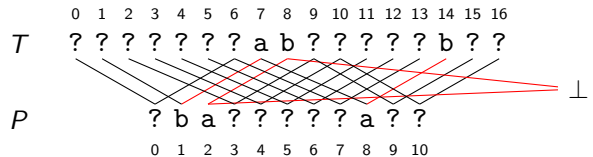

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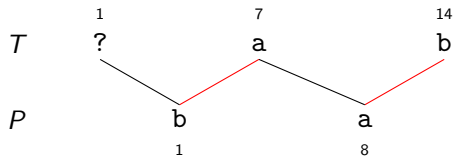

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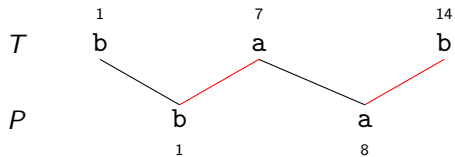

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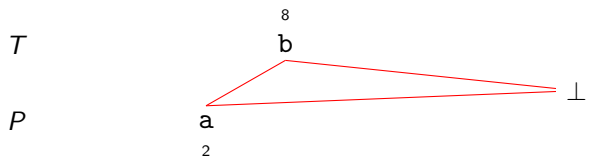

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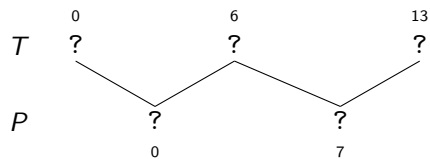

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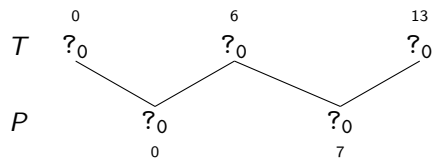

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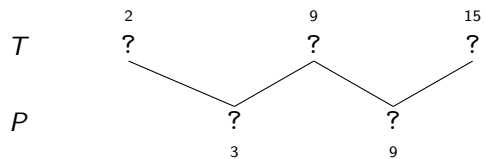

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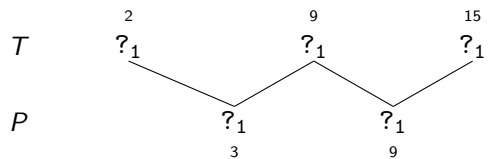

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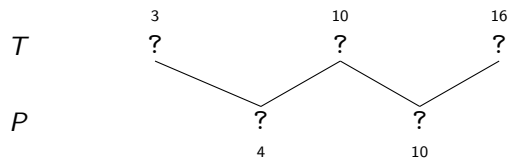

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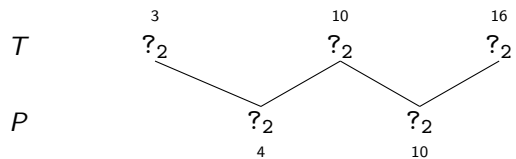

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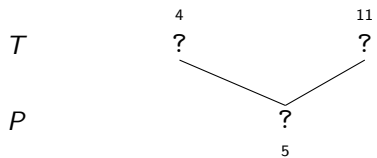

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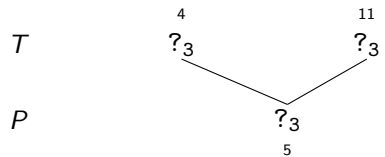

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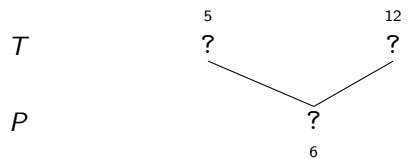

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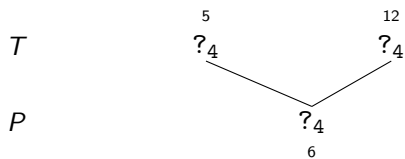

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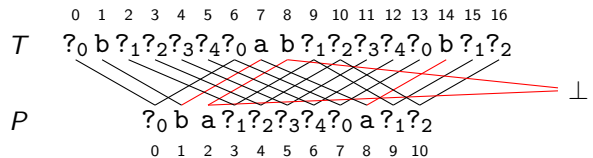

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T_S $?_0$ $?_1?_2?_3?_4?_0$ $?_1?_2?_3?_4?_0$ $?_1?_2$

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1. Bob receives $S = \{\mathcal{A}_1, \mathcal{A}_2\}$.
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$$T_S = \quad ?_0?_1?_2?_3?_4?_0?_1?_2?_3?_4?_0?_1?_2$$

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$$T_S = \quad \mathbf{0\,1\,2\,4\,0\,1\,2\,3\,4\,0\,1\,2}$$

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T_S **0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5**

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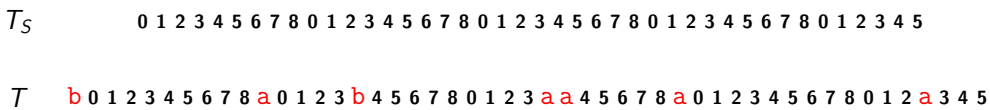

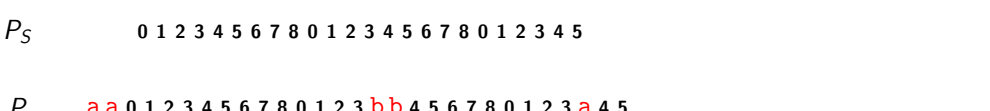

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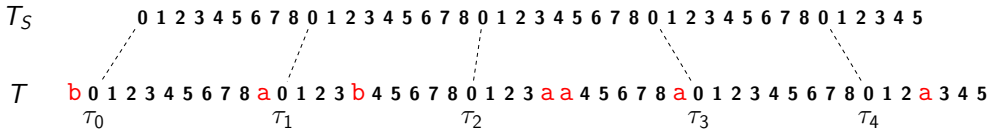

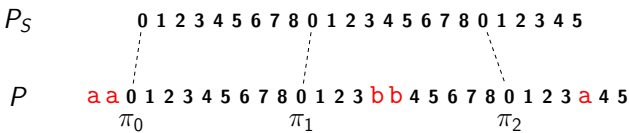

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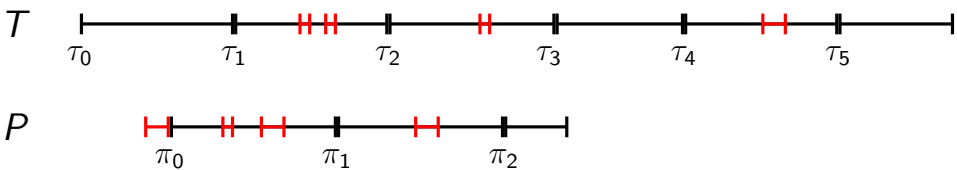

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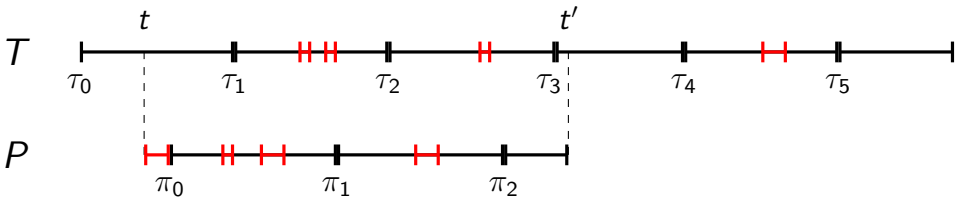

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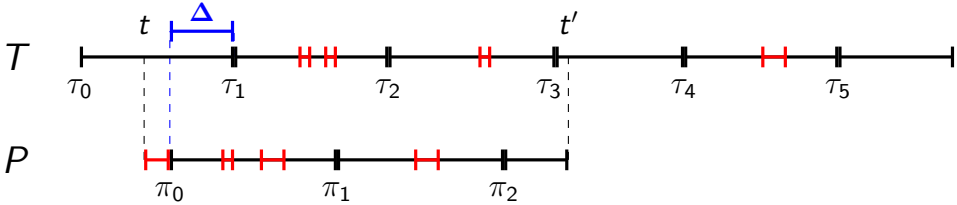

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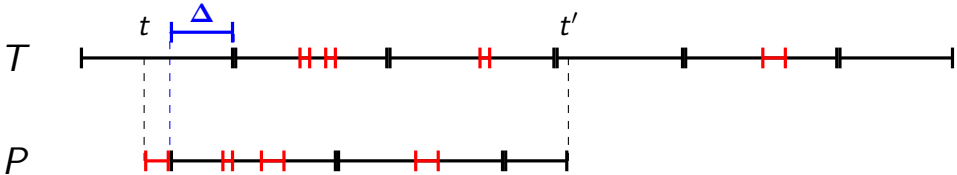

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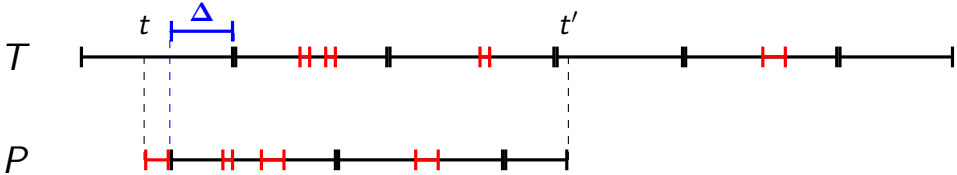

Case $\boldsymbol{\Delta} \gg k$ (large) $\Longrightarrow$ if $\mathcal{X}$ is added to S, the # of black components at least halves.

N.B. We can assume no such alignments exist. Communication cost $= \mathcal{O}(\underbrace{|S|}_{\leq \log m} \cdot k \cdot \log(m|\Sigma|))$

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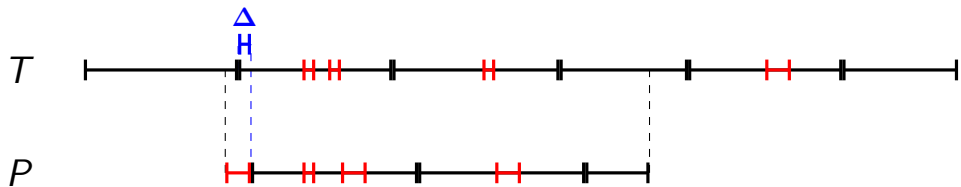

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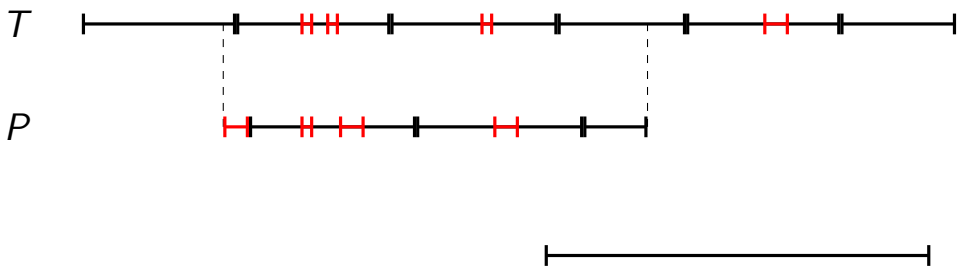

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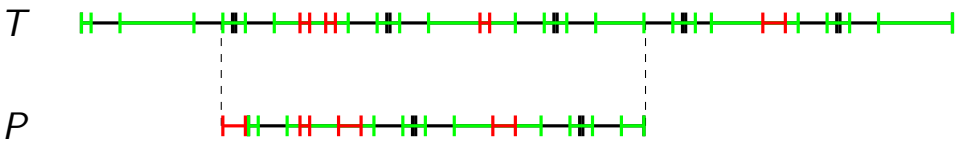

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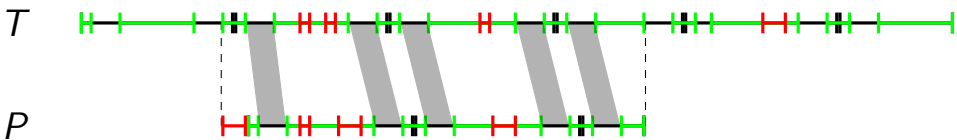

encoding cost $\approx$ same cost of what we encoded in 1.

Case $\boldsymbol{\Delta} \lesssim k$ (small) $\Longrightarrow$ an optimal alignment $P \rightsquigarrow T[t..t']$ matches characters in the same uncovered black components.

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Key intuition:

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Challenge 2: Arguments become very tricky because some cyclic dependencies.

Improved Construction (Intuition)

Key intuition:

While adding "uncaptured" alignments, we can do better than adding the whole alignments:

When P_S has 2^i period occurrences we can add a subset of the alignment of cost $\approx k/2^i$.

Challenge 1: Now S contains alignments of fragments of P onto fragments on T.

Challenge 2: Arguments become very tricky because some cyclic dependencies.

Challenge 3: For each alignment we send $\Omega(\log m)$ bits, this is a problem when S grows to size $\omega(k)$. When this happens we: P and T are periodic in a stronger sense (in the sense of [CKW20]), and that in such case we can construct S of size 3.

Thanks!